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Email address : CJCMEmail@gmail.com
Postal address : P.O .Box 44815 -1165 (Dr.S.M.Ali Raeisian), Roodsar, Gilan , Iran
Phone : (+98) 13-42 613646 , (+98) 911 143 4988

Editing & Paging : Rah-poyan Andisheh & Honar cultural & artistic institute.
rpandishehonar@gmail.com

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Defect Numbers of the Dirichlet Problem for Higher Order Partial Differential Equations in the Unit Disc

A. H. Babayan, V.A. Babayan
National Polytechnic University of Armenia

E-mail: barmenak@gmail.com, bvazgen@gmail.com

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Abstract: We consider the Dirichlet problem in the unit disc for the linear partial differential equation with constant coefficients. The formulas for the determination the defect numbers of the problem were found, and for the improperly equation was determined the functional class, were this problem is Notherian.

Let D be a unit disk of the complex plane and $\Gamma = \partial D$. We consider an elliptic equation

$$\sum_{k=0}^{2N} A_k \frac{\partial^{2N} U}{\partial x^k \partial y^{2N-k}} = 0, \quad (x,y) \in D, \tag{1}$$

where A_k are complex constants $(A_0 \neq 0)$, such that the roots λ_j, $j = 1,\ldots,2N$ of the characteristic equation

$$\sum_{k=0}^{2N} A_k \lambda^{2N-k} = 0, \tag{2}$$

satisfy the conditions

$$\Im\lambda_k > 0, k = 1,\ldots,P, \ \Im\lambda_k < 0, \ k = P+1,\ldots,2N. \tag{3}$$

The solution U of the equation (1) (in the class $C^{2N}(D) \cap C^{(N-1,\alpha)}(D \cup \Gamma)$) satisfies Dirichlet conditions:

$$\left.\frac{\partial^k U}{\partial r^k}\right|_{\Gamma} = f_k(x,y), \quad (x,y) \in \Gamma, \ k = 0,\ldots,N-1 \tag{4}$$

on the boundary Γ.

Here $f_k \in C^{(N-k-1,\alpha)}(\Gamma)$, $(k=0,\ldots,N-1)$ are given functions on Γ, $\frac{\partial}{\partial r}$ is the derivative with respect to the inner normal to Γ.

The cases $P=N$ (the equation (1) is properly elliptic) and $P \neq N$ when the equation (1) is improperly elliptic, are sufficiently different. If the equation (1) is properly elliptic, then the problem (1),(4) is Fredholmian (see [Tovmasyan 1998], [Lions, Madgenes 1968]). In this case we want to define necessary and sufficient unique solvability conditions and the defect numbers of the problem if these conditions fail. In this paper we denote by defect numbers the numbers of linearly independent solutions of homogeneous problem (when $f_k \equiv 0$ for all k) and the number of linearly independent conditions to the boundary functions f_k, when inhomogeneous problem has a solution. For improperly elliptic equation (1) the Dirichlet problem (1), (4) (as all classical boundary value problems) is neither Fredholmian nor Notherian (see [Bitcadze 1961]). More precisely, this problem is not normal solvable, and for some values of coefficients, this problem may have infinite defect numbers. In this case we want to define the solutions of homogeneous problem and the class of boundary functions for which the inhomogeneous problem has a solution. In the paper we present some results for the problem (1), (4) in case of elliptic equation (1), and some considerations about homogeneous problem for not elliptic equation (1) (when some roots λ_j of characteristic equation (2) may be real).

We start from the elliptic case. In the paper [Tovmasyan 1969] complete research of the Dirichlet problem for the system of second order elliptic equations with constant coefficients in elliptic domains was conducted. The defect numbers were defined in explicit form. The difference between weakly and strongly connected elliptic systems was found. It was shown that the defect numbers are finite and the problem is Fredholmian for weakly connected systems and neither Fredholmian nor Notherian for strongly connected system. The second order improperly elliptic equation: the equation (1) for $N=1$, if the roots of equation (2) satisfy the conditions $\Im\lambda_1 > 0$, $\Im\lambda_2 > 0$ (or strongly connected system of 2nd order) was studied in [Tovmasyan 1968]. The Dirichlet problem for this equation in the unit disc was completely researched. The conditions on the coefficient of the equation (1), for which the homogeneous Dirichlet problem (1), (4) has infinite number of linearly independent solutions, were found. In the first time was presented the functional class where this problem is Notherian. The formula of the general solution of the equation (1) makes it easier to research the problem in the unit disc and the representation of this solution on the unit circumference was also found in this work.

Lemma. [Tovmasyan 1968]. *Let* $|\mu|<1$ *and function* Φ *is analytic in the domain* $D(\mu)=\{z+\mu\overline{z}:|z|<1\}$ *and is from the class* $C^{(\alpha)}\left(\overline{D(\mu)}\right)$*. Then for* $|z|=1$ *the function* $\Phi(z+\mu\overline{z})$ *may be represented in the form*

$$\Phi(z+\mu\overline{z})=\phi(z)+\phi(\mu\overline{z}),\quad \phi(z)=\sum_{n=0}^{\infty}D_k z^k. \tag{5}$$

The function Φ *is uniquely determined by the function* ϕ*:*

$$\Phi(\zeta)=\phi\left(\frac{\zeta+\sqrt{\zeta^2-4\mu}}{2}\right)+\phi\left(\frac{\zeta-\sqrt{\zeta^2-4\mu}}{2}\right) \tag{6}$$

Here $\zeta\in D(\mu)$ *and we get a branch of the root, for which* $\zeta^{-1}\sqrt{\zeta^2-4\mu}\to 1$ *for* $\zeta\to\infty$.

Using this lemma, it is possible to reduce the problem (1), (4) to algebraic problems. To explain this we consider the simplest case - $N=1, P=1$. Let's denote

$$\mu_1=\frac{i-\lambda_1}{i+\lambda_1},\ \ \mu_2=\frac{i+\lambda_1}{i-\lambda_2}, \tag{7}$$

where λ_j are the roots of the equation (2). Then the solution of the equation (1) may be represented in the form

$$u(x,y)=\Phi_1(z+\mu_1\overline{z})+\Phi_2(\overline{z}+\mu_2 z),\ \ z=x+iy,\ \ (x,y)\in D, \tag{8}$$

where μ_j are constants (7) (taking into account the condition $\Im\lambda_1>0>\Im\lambda_2$, we have $|\mu_j|<1$), and the unknown functions Φ_j, $j=1,2$ are analytic in the domains $D(\mu_1)=\{z+\mu_1\overline{z}:|z|<1\}$ and $D(\mu_2)=\{\overline{z}+\mu_2 z:|z|<1\}$ correspondingly. Substituting the representation (8) in the boundary condition (4) (for $N=1, P=1$ it remains the only condition) we get

$$\Phi_1\left(e^{i\theta}+\mu_1 e^{-i\theta}\right)+\Phi_2\left(e^{-i\theta}+\mu_2 e^{i\theta}\right)=f_0(\theta),\quad 0\le\theta<2\pi. \tag{9}$$

Now, using lemma, represent the functions Φ_j, $j=1,2$ via functions, analytic in the unit disc:

$$\begin{aligned}\Phi_1\left(e^{i\theta}+\mu_1 e^{-i\theta}\right)&=\varphi_1\left(e^{i\theta}\right)+\varphi_1\left(\mu_1 e^{-i\theta}\right),\\ \Phi_2\left(e^{-i\theta}+\mu_2 e^{i\theta}\right)&=\varphi_2\left(e^{-i\theta}\right)+\varphi_2\left(\mu_2 e^{i\theta}\right),\qquad 0\le\theta<2\pi\end{aligned} \tag{10}$$

The functions φ_j are analytic in the unit disc, and, therefore, may be expanded via the Taylor series; since the function f_0 belongs to the class of Holder continuous functions it may be represented via the Fourier series. We get

$$\varphi_j(z)=\sum_{k=0}^{\infty}A_{jk}z^k, \quad |z|\le 1, \quad j=1,2; \quad f_0(\theta)=\sum_{k=-\infty}^{\infty}d_k e^{ik\theta}. \tag{11}$$

Let's substitute the representations (10) and (11) in the boundary condition (9) :

$$\sum_{k=0}^{\infty}A_{1k}e^{ik\theta}+\sum_{k=0}^{\infty}A_{1k}\mu_1^k e^{-ik\theta}+\sum_{k=0}^{\infty}A_{2k}e^{-ik\theta}+\sum_{k=0}^{\infty}A_{2k}\mu_2^k e^{ik\theta}$$
$$=\sum_{k=-\infty}^{\infty}d_k e^{ik\theta}, \qquad 0\le\theta<2\pi$$

This equality holds for all points of the unit circumference, therefore, coefficients of corresponding powers of $e^{ik\theta}$, $k=0,\pm1,\pm2,\ldots$ from the left and right parts of the equality must be the same. We get

$$\begin{cases} A_{1k}+A_{2k}\mu_2^k=d_k \\ A_{1k}\mu_1^k+A_{2k}=d_{-k} \end{cases}, \ k\ge 1; \quad 2A_{10}+2A_{20}=d_0 . \tag{12}$$

Thus, the problem (1), (4) is reduced to the systems (12). The determinant of the main matrix of the system (12) is equal

$$\Delta_k=\begin{vmatrix} 1 & \mu_2^k \\ \mu_1^k & 1 \end{vmatrix}=1-(\mu_1\mu_2)^k\ne 0, \ k=1,2,\ldots . \tag{13}$$

Since $|\mu_j|<1$ these determinants are not equal to zero. So, we determine the coefficients A_{1k}, A_{2k} for $k\ge 1$ uniquely. The determinants Δ_k tend to one for $k\to\infty$, hence the corresponding functions φ_j and the boundary function f_0 belong to the same functional class. After determination of the functions φ_j, using lemma, we get the functions Φ_j and therefore, uniquely determine the solution of the corresponding problem (1), (4). We cannot determine the coefficients A_{10}, A_{20} because we have only the sum of these coefficients, but it is enough for the uniqueness of the problem. Thus, the Dirichlet problem (1), (4) for the properly elliptic equation (1) is uniquely solvable (this result was proved earlier, using another method, see, for example, [Lions, Madgenes 1968]).

Let's consider the case of the second order improperly elliptic equation - $N=2,\ P=2$. We suppose, that $\lambda_1 \neq \lambda_2,\ \Im\lambda_j > 0,\ \lambda_j \neq \pm i$. In this case, denoting

$$\mu_j = \frac{i-\lambda_j}{i+\lambda_j},\ \ j=1,2, \tag{14}$$

we present the general solution of the equation (1) in the form:

$$u(x,y) = \Phi_1(z+\mu_1\bar{z}) + \Phi_2(z+\mu_2\bar{z}),\quad z=x+iy,\ \ (x,y)\in D \tag{15}$$

Now, using the considerations, analogous to the case of properly elliptic equation (1), we reduce the problem (1), (4) to the system:

$$\begin{cases} A_{1k}+A_{2k}=d_k \\ A_{1k}\mu_1^k + A_{2k}\mu_2^k = d_{-k} \end{cases},\ k\geq 1;\ \ 2A_{10}+2A_{20}=d_0\ . \tag{16}$$

In this case the determinant of the system has following form:

$$\Delta_k = \begin{vmatrix} 1 & 1 \\ \mu_1^k & \mu_2^k \end{vmatrix} = \mu_2^k - \mu_1^k\ ,$$

Therefore we have the following result:

Theorem [Tovmasyan 1968]. *We consider (1), (4) problem for* $N=P=2$. *We suppose, that* $\lambda_1 \neq \lambda_2$, $\lambda_j \neq \pm i$, $\Im\lambda_j > 0$. *Let's define* μ_j *by the formulas (14). Without loss of generality we suppose that* $|\mu_2| \leq |\mu_1|$ *and introduce* $A_{1,1}(|\mu_1|)$ *- class of functions, analytic in the ring* $|\mu_1| < |z| < 1$ *and satisfying Holder condition in the closed ring* $|\mu_1| \leq |z| \leq 1$. *Then*

1) *If* $\mu_1 = \exp(2\pi imn^{-1})\mu_2$ *for integers* m *and* n, *then the homogeneous problem has infinitely many linearly independent solutions and for the solvability of inhomogeneous problem it is necessary for the boundary function* f_0 *to satisfy infinitely many linearly independent conditions.*

2) *If* $(2\pi)^{-1}\arg\mu_1\mu_2^{-1}$ *is an irrational number, or* $|\mu_1| \neq |\mu_2|$ *then the homogeneous problem has a unique solution for arbitrary boundary function from the class* $A_{1,1}(|\mu_1|)$.

Other cases of the location of the roots in this paper were considered too. And it must be mentioned that the functional class from this article, where classical

boundary value problems for improperly elliptic equations may be considered, was defined for the first time.
After that investigations were continued for properly elliptic equation (1).
In [Soldatov 2005] the unique solvability of the Dirichlet problem for the second order elliptic system was investigated. For the fourth order properly elliptic equation $N = 2, P = 2$ the necessary and sufficient conditions for the (1), (4) problem's unique solvability were found in [Babayan 1999] and [Buryachenko 2000]. The analogous result for higher order equation (if the characteristic equation (2) has only simple roots) was achieved in [Babayan 2004], [Tovmasyan 2002], [Burskii, Buryachenko 2005]. The properly elliptic equation, when the multiplicity of the roots of equation (2) is not greater than two, was considered in [Irician 2003]. If the unique solvability fails, in [Babayan 1999], [Babayan 2004], [Tovmasyan 2002], [Irician 2003] the formulas for the determination of the defect numbers of (1), (4) by the coefficients of (1) were found. In [Tovmasyan 2002] the problem (1), (4) was researched in arbitrary multiply connected domain (it was reduced to a second order Fredholm equation). In all these articles it was supposed, that multiplicity of the roots of the equation (2) is not greater than two. The problem was, that the known formulas for representation of the general solution of the equation (1) ([Bicadze 1966], [Vekua 1948], [Tovmasyan 1998]) do not «work well» in the case of higher multiplicity of the roots of characteristic equation (2).
In [Babayan 2003] the representation of the general solution of equation (1), which is appropriate for the cases of simple and multiple roots, was found.

General solution of the equation (1)

Let us represent the equation (1) in the complex form, using operators of complex differentiation

$$z = x + iy,\ \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The equation (1) is reduced to the form

$$\prod_{k=1}^{p}\left(\frac{\partial}{\partial \overline{z}} - \mu_k\frac{\partial}{\partial z}\right)^{l_k}\prod_{j=1}^{q}\left(\frac{\partial}{\partial z} - \nu_j\frac{\partial}{\partial \overline{z}}\right)^{m_j} U = 0, \tag{17}$$

where

$$\mu_j = \frac{i - \lambda_j}{i + \lambda_j} \text{ for } \Im\lambda_j > 0 \text{ and } \nu_j = \frac{i + \lambda_j}{i - \lambda_j} \text{ for } \Im\lambda_j < 0 \tag{18}$$

$$\sum_{k=1}^{p} l_k = P,\ \sum_{j=1}^{q} m_j = 2N - P,\ |\mu_k| < 1,\ k = \overline{1,p},\ |\nu_j| < 1,\ j = \overline{1,q}\,.$$

The general solution of (1) may be represented in the form:

$$U(x,y)=\sum_{k=1}^{p}\sum_{a=0}^{l_k-1}\left(\frac{\partial}{\partial\phi}\right)^a\Phi_{ka}\left(z+\mu_k\overline{z}\right)+\sum_{j=1}^{q}\sum_{b=0}^{m_j-1}\left(\frac{\partial}{\partial\phi}\right)^b\Psi_{kb}\left(\overline{z}+\overline{\nu}_k z\right)$$

(19)

Using these notions, the unique solvability conditions of the problem (1), (4) for properly elliptic equation (1) may be written in following form [Babayan, 2003]. If μ_k has the multiplicity l_k, and ν_j has the multiplicity m_j, then we define the N - dimensional vectors

$$a_k^{s+1}=\left(C_{N-1}^{s}\mu_k^{N-s-1},\ldots,C_{s+1}^{s}\mu_k,C_s^s,0,\ldots,0\right)^T, \qquad (20)$$

$$b_j^{t+1}=\left(0,\ldots,0,C_t^t,C_{t+1}^{t}\nu_j,\ldots,C_{N-1}^{t}\nu_j^{N-t-1}\right)^T, \qquad (21)$$

where $0\le s\le l_k-1,\ k=1,\ldots,p$ and $0\le t\le m_j-1,\ j=1,\ldots,q$. Let A and B be a square matrices of order N

$$A=\left(a_1^1\ldots a_1^{l_1}a_2^1\ldots a_2^{l_2}\ldots a_p^1\ldots a_p^{l_p}\right),\ B=\left(b_1^1\ldots b_1^{m_1}\ldots b_q^1\ldots b_q^{m_q}\right), \qquad (22)$$

and M, H are Jordan matrices

$$M=diag\left(J_{l_1}(\mu_1)\ldots J_{l_p}(\mu_p)\right),\ H=diag\left(J_{m_1}(\nu_1)\ldots J_{m_q}(\nu_q)\right). \qquad (23)$$

Here $J_k(\lambda)$ is a Jordan block of order k with diagonal elements λ.
The problem (1), (4) is uniquely solvable if and only if the matrix

$$\Omega_l=\begin{pmatrix}A & BH^l\\ AM^l & B\end{pmatrix} \qquad (24)$$

is non-singular for $l=N+1,N+2,\ldots$, i.e.

$$\Delta_l=det\Omega_l\neq 0,\quad l=N+1,N+2,\ldots. \qquad (25)$$

If for some $k_0\ge N+1$ we get $\Delta_{k_0}=0$, then the homogeneous problem has a solution, which is a polynomial of order $N+k_0-1$. The boundary functions in this case must satisfy one orthogonality condition for the solvability of inhomogeneous problem. Therefore the defect numbers of the problem (1), (4) will be determined by the formula:

$$K_1=K_2=\sum_{l=N+1}^{\infty}\left(2N-rank\Omega_l\right) \qquad (26)$$

In these formulas it was supposed, that all roots are not equal $\pm i$. This case was considered earlier ([Vekua, 1948], [Tovmasyan, 1998]). In our notations, if some root is equal $\pm i$ with multiplicity k then in the matrix Ω_i matrix column $\begin{pmatrix} I_k & 0 \end{pmatrix}^T$ (where I_k is a k order unit matrix) must be added from the left or matrix column $\begin{pmatrix} 0 & I_k \end{pmatrix}^T$ from the right correspondingly.

Example. For $N = 2$ in the case of properly elliptic equation (1), supposing $\lambda_1 \neq \lambda_2$, $\Im\lambda_j > 0,\ j = 1,2$ $\lambda_3 \neq \lambda_4$, $\Im\lambda_j < 0,\ j = 3,4$, we determine μ_j and ν_j by the formulas (18) and the matrix (24) Ω_k has the form:

$$\Omega_k = \begin{pmatrix} 1 & 1 & \nu_1^{k+1} & \nu_2^{k+1} \\ \mu_1 & \mu_2 & \nu_1^k & \nu_2^k \\ \mu_1^k & \mu_2^k & \nu_1 & \nu_2 \\ \mu_1^{k+1} & \mu_2^{k+1} & 1 & 1 \end{pmatrix}, \quad k = 3, 4, \ldots \tag{27}$$

and in (27)

$$A = \begin{pmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{pmatrix}, \quad B = \begin{pmatrix} \nu_1 & \nu_2 \\ 1 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad H = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}.$$

The case of improperly elliptic equation may be considered in the same way. In [Babayan 2007] fourth order equation (1) was investigated. In this article it was shown that the different locations of the roots reduce to different results about solvability of the problem (1), (4). As an example, let's formulate one of the obtained results.

Theorem. *We consider the case* $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_j \neq i,\ \Im\lambda_j > 0$, *for* $j = 1,2,3$ *;* $\lambda_4 \neq -i$, $\Im\lambda_4 < 0$. *In this case equation (1), using notions (18) is reduced to the form:*

$$\left(\frac{\partial}{\partial \bar{z}} - \mu_1 \frac{\partial}{\partial z}\right)^2 \left(\frac{\partial}{\partial \bar{z}} - \mu_2 \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z} - \nu \frac{\partial}{\partial \bar{z}}\right) U = 0, \quad (x, y) \in D . \tag{28}$$

We represent the Dirichlet boundary conditions in equivalent form:

$$\left.\frac{\partial u}{\partial \bar{z}}\right|_\Gamma = F_1(x, y), \quad \left.\frac{\partial u}{\partial z}\right|_\Gamma = F_2(x, y), \quad (x, y) \in \Gamma; \ u(1,0) = f(1,0) \tag{29}$$

Here $F_1(x,y)=\frac{z}{2}\left(g(x,y)+i\frac{\partial f}{\partial\varphi}(x,y)\right),$

$$F_2(x,y)=\frac{\overline{z}}{2}\left(g(x,y)-i\frac{\partial f}{\partial\varphi}(x,y)\right),\ z=re^{i\varphi}\in\Gamma. \tag{30}$$

Let's denote

$$\Delta_k(\mu_1,\mu_2,\nu)=\det\begin{pmatrix}1 & 0 & 1 & \nu^{k+1}\\ \mu_1 & 1 & \mu_2 & \nu^k\\ \mu_1^k & k\mu_1^{k-1} & \mu_2^k & \nu\\ \mu_1^{k+1} & (k+1)\mu_1^k & \mu_2^{k+1} & 1\end{pmatrix}.$$

$$\equiv \det\Omega_k,\quad k=3,4,\ldots$$

Then the homogeneous problem (28), (29) has a finite number of linearly independent solutions, which are defined by the formula

$$N_0=\sum_{k=3}^{\infty}\left(4-rank\Omega_k\right). \tag{31}$$

For the solvability of the inhomogeneous problem it is necessary for the functions (29) F_j ($j=1,2$) to be analytic in the ring $\sigma<|z|<1$ and sufficient, that these functions satisfy Hölder condition in the closed ring $\sigma\le|z|\le1$ with their first order derivatives.

It must be mentioned, that in case of fourth order improperly elliptic equation (1) an appropriate set of boundary functions for the normal solvability of the problem (1), (4) is the set $A_{2,2}(\sigma)$ of functions analytic in the ring $\sigma<|z|<1$ and with first order derivatives satisfying Holder condition in the closed ring $\sigma\le|z|\le1$ (compare with the set $A_{1,1}(|\mu_1|)$ from [Tovmasyan 1968]).

Unique solvability of the Dirichlet problem was considered not only for elliptic equation (1). In the papers [Bourgin, Duffin 1931], [John 1941], [Alexandryan 1960], [Hovsepyan 1969] the conditions of nontrivial solvability of the homogeneous problem for the second order hyperbolic equation in the different domains were found. Further, in the [Burskii 2002] "equation-domain duality relation" was proved which in our case has the form:

$$L(D_x)\left[\left(1-|x|^2\right)^N u(x)\right]=0,\quad (\Delta+1)^N\left[L(\xi)v(\xi)\right]=0\,.$$

It was shown that every nontrivial polynomial solution of the second equation corresponds to the nontrivial polynomial solution of the original equation. Using this result the Dirichlet problem nontrivial solvability question for arbitrary (may be not elliptic) equation (1) was investigated in [Buryachenko 2000], [Burskii 2002], [Burskii, Buryachenko 2005], [Buryachenko 2010].
We consider the problem (1), (4) in the unit disc. In all cases we get, the homogeneous problem may have polynomial solutions only. The analogous results may be proved in elliptic domains too. In some sense this shape of domain is important. To see that, we must mention the article [Chamberland, Siegel 2000] were was proved, that the problem

$$\Delta u = 0,\ (x,y)\in\mathbb{R}^2;\ u(x,y) = f(x,y),\ \text{on } \psi(x,y) = 0,$$

where f,ψ are polynomials, has the polynomial solution u, such that $\deg u \le \deg f$ for arbitrary polynomial f only in the case $\deg\psi \le 2$, that is, the boundary of the considered domain must be the second order curve.
Thus, we get the formulas (26) for the determination of the defect numbers of the Dirichlet problem for properly elliptic equation (1). The sum in this formula is finite, because $\det\Omega_k \to c \ne 0$ for $k \to \infty$, and therefore, $rank\Omega_k = 2N$ for sufficiently big k. A question can be asked here: What value may have the defect numbers K_1 and K_2? Are they big or small? We can answer to this question for some cases of the fourth order equation (1).
Let's consider fourth order properly elliptic equation (1) when the roots of the characteristic equation (2) satisfy the condition:

$$\lambda_1 = i,\ \lambda_2 \ne i,\ \Im\lambda_2 > 0;\ \Im\lambda_j < 0,\ \ j = 3,4.$$

In this case, using the notations (18), the equation (1) may be represented in the form:

$$\frac{\partial}{\partial\bar{z}}\left(\frac{\partial}{\partial\bar{z}} - \mu_2\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial z} - \nu_1\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z} - \nu_2\frac{\partial}{\partial\bar{z}}\right)U = 0,\ \ (x,y)\in D, \tag{32}$$

were $|\mu_2| < 1,\ |\nu_j| < 1$ for $j = 1,2$. Let's denote $\delta = \mu_2\nu_1,\ \gamma = \mu_2\nu_2$. The following theorem was proved.
Theorem. **[Babayan 1999].** *The Dirichlet problem (32), (4) is uniquely solvable if and only if one of the following conditions is satisfied:*

1) $\mu_2 = 0$,

2) $\mu_2 \ne 0,\ \nu_1 = \nu_2$ *and*

$$\Delta_{1,k} = \sum_{j=0}^{k-2} j+1\ \delta^j \ne 0, \qquad k = 3,4,\ldots, \tag{33}$$

3) $\mu_2 \neq 0, \quad \nu_1 \neq \nu_2$ *and*

$$\Delta_{2,k} = \sum_{j=0}^{k-2}\sum_{i=0}^{j}\delta^i\gamma^{j-i} \neq 0, \quad k = 3,4,\dots \ . \tag{34}$$

If the conditions (33) (or (34)) failed for some number k_0, then the homogeneous problem (32), (4) has one solution – polynomial of order k_0+1, and for the solvability of the corresponding inhomogeneous problem it is necessary for the boundary functions to satisfy one solvability condition (for different k_0 these conditions are linearly independent). Thus the defect numbers of the problem (32), (4) is the quantity of the $\Delta_{1,k}$ (or $\Delta_{2,k}$) which are equal to zero.

Later this result was refined. In [Babayan 2011] it was proved, that if $\gamma = r\delta$ where r is the real number (or what is equivalent $\nu_2 = r\nu_1$), then the condition (34) may fail only for one number k_0, therefore, the defect numbers of the Dirichlet problem (32), (4) may only have two values in this case: zero, when the problem is uniquely solvable ((34) holds), and one if (34) failed. The same result was proved for the case $\mu_2 \neq 0, \ \nu_1 = \nu_2$ in [Babayan 2012].

In [Babayan 2015] we consider the case when fourth order equation (1) has double roots.

Supposing, that the roots of characteristic equation λ_j ($j = 1,2,3,4$) satisfy the condition

$$\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4, \ \lambda_j \neq \pm i, \ \ j = 1,2,3,4, \tag{35}$$

we separately research the following three cases: 1) $\Im\lambda_1 > 0 > \Im\lambda_3$ – the equation (1) is properly elliptic, 2) $\Im\lambda_1 \geq \Im\lambda_3 > 0$ – the equation (1) is improperly elliptic, and the last case 3) if one of the roots is real – the equation (11) is not elliptic.

First, using the operators of complex differentiation we represent the equation (1) in the complex form.

1). The case of properly elliptic equation. Using operators of complex differentiation, we represent the equation (1) in the form:

$$\left(\frac{\partial}{\partial \overline{z}} - \mu\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial z} - \nu\frac{\partial}{\partial \overline{z}}\right)^2 u(x,y) = 0 \ . \tag{36}$$

Here from (18) $\mu = \dfrac{i-\lambda_1}{i+\lambda_1}$, $\nu = \dfrac{i+\lambda_3}{i-\lambda_3}$, and, therefore, from the conditions $\Im\lambda_1 > 0 > \Im\lambda_3$ and (35) we have

$$|\mu| < 1, \ |\nu| < 1, \quad \mu\nu \neq 0 \ . \tag{37}$$

We represent the conditions (4) in equivalent form (29). We suppose in this case $f \in C^{(1,\alpha)}(\Gamma)$, $g \in C^{(\alpha)}(\Gamma)$, therefore, F_j belongs to the space $C^{(\alpha)}(\Gamma)$ for $j = 1, 2$.

Theorem 1. ***1*** *Let's denote* $z = \mu\nu$ *and* $t = 0.5\left(z^{0.5} + z^{-0.5}\right)$. *Then the Dirichlet problem (36), (4) is uniquely solvable if and only if*

$$U_{k-1}^2(t) \neq k^2, \quad k = 3, 4, \ldots, \tag{38}$$

where U_{k-1} is the second kind Tchebychev polynomial of order $k-1$. If the conditions (38) fail for any k_0, then the homogeneous problem (38), (4) has a nontrivial solution, which is a polynomial of order $k_0 + 1$. In this case one linearly independent condition on the functions F_j is necessary for the solvability of the inhomogeneous problem (36), (4). Therefore the defect numbers of the problem (36), (4) are equal to quantity of numbers for which the condition (38) failed.

2). Improperly elliptic equation. In this case we have $\Im\lambda_1 \geq \Im\lambda_3 > 0$, therefore, an equation (1) may be represented in the form:

$$\left(\frac{\partial}{\partial \overline{z}} - \mu_1 \frac{\partial}{\partial z}\right)^2 \left(\frac{\partial}{\partial \overline{z}} - \mu_2 \frac{\partial}{\partial z}\right)^2 u(x, y) = 0, \tag{39}$$

where $\mu_1 = \dfrac{i - \lambda_1}{i + \lambda_1}$, $\mu_2 = \dfrac{i - \lambda_3}{i + \lambda_3}$. We get:

$$\mu_1 \neq \mu_2, \quad |\mu_1| < 1, \quad |\mu_2| < 1. \tag{40}$$

Let's define the functional class, which is necessary for further considerations

Definition 2 *We denote* $B^{(m,\alpha)}(\delta)$ *the class of functions analytic in* $R = \{z : \delta < |z| < 1\}$, *which belong to the class* $C^{(\alpha)}(\overline{R})$ *(i.e. satisfy the Hölder condition in the closure of* R *) with their derivatives of order not more than* m.
The following statement is proved.

Theorem 2. 3 *We suppose that* $|\mu_1| \geq |\mu_2|$ *and denote* $z = \mu_2\mu_1^{-1}$ *and* $t = 0.5\left(z^{0.5} + z^{-0.5}\right)$. *Let the boundary functions in (29)* F_j *for* $j = 1, 2$ *belong to the class* $B^{(1,\alpha)}(|\mu_1|)$. *Then, if the conditions (38) hold, then the problem (39), (4) is uniquely solvable. If the conditions (38) fail for any* k_0, *then the homogeneous* *problem (39), (4) has a nontrivial solution, which is a polynomial of order* k_0+1. *In this case one linearly independent condition on the functions* F_j *is necessary for the solvability of the inhomogeneous problem (39), (4). Therefore, the defect numbers of the problem (39), (4) are equal to the quantity of numbers for which the condition (36) failed.*

3). Non-elliptic equation.

Theorem 3. 4 *Let the numbers* μ_1, μ_2, z, t *be defined as in theorem 2. Then the homogeneous problem (39), (4) has no nontrivial solutions if and only if the conditions (36) hold.*

Example. Let's consider an equation (36). Expanding brackets, we get:

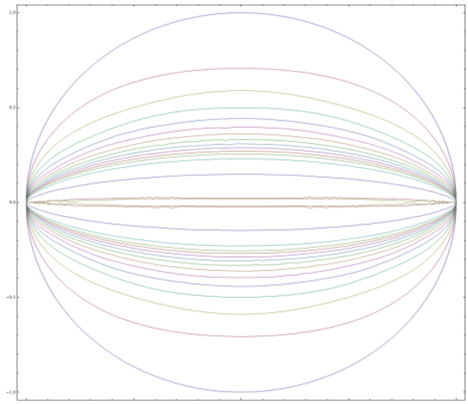
$$\left(\frac{\partial^4}{\partial\overline{z}^2\partial z^2}\left(1-4\mu\nu+\mu^2\nu^2\right)-2\mu\frac{\partial^4}{\partial z\partial\overline{z}^3}\left(1+\mu\nu\right)-2\nu\frac{\partial^4}{\partial\overline{z}^3\partial z}\left(1+\mu\nu\right)+\nu^2\frac{\partial^4}{\partial\overline{z}^4}+\mu^2\frac{\partial^4}{\partial z^4}\right)u\left(x,y\right)=0\ ,$$

It is easy to verify, that the function $U\left(x,y\right)=\left(1-z\overline{z}\right)^2$ is a solution of (36) if and only if

$$1-4\mu\nu+\mu^2\nu^2=0\Leftrightarrow 1-4z+z^2=0\Leftrightarrow U_2^2\left(t\right)-3^2=0\,.$$

On the boundary Γ this function vanishes. Thus, if the condition (38) fails for $k=3$ then the fourth order polynomial U is a nontrivial solution of the homogeneous problem (36), (4).

Now, let's draw the graph of the curves L_k defined by the equations $\left|U_{k-1}(x,y)\right|=k$. We draw it in the unit disc ($\left(x,y\right)\in D$).



ContourPlot [{Abs[ChebyshevU[1, x + Sqrt[-1]*y]] == 2, Abs[ChebyshevU[2, x + Sqrt[-1]*y]] == 3, Abs[ChebyshevU[3, x + Sqrt[-1]*y]] == 4, Abs[ChebyshevU[4, x + Sqrt[-1]*y]] == 5,Abs[ChebyshevU[5, x + Sqrt[-1]*y]] == 6, Abs[ChebyshevU[6, x + Sqrt[-1]*y]] == 7, Abs[ChebyshevU[7, x + Sqrt[-1]*y]] == 8, Abs[ChebyshevU[8, x + Sqrt[-1]*y]] == 9, Abs[ChebyshevU[9, x + Sqrt[-1]*y]] == 10, Abs[ChebyshevU[10, x + Sqrt[-1]*y]] == 11,Abs[ChebyshevU[11, x + Sqrt[-1]*y]] == 12, Abs[ChebyshevU[12, x + Sqrt[-1]*y]] == 13, Abs[ChebyshevU[14, x + Sqrt[-1]*y]] == 15, Abs[ChebyshevU[26, x + Sqrt[-1]*y]] == 27, Abs[ChebyshevU[200, x + Sqrt[-1]*y]] == 201, Abs[ChebyshevU[300, x + Sqrt[-1]*y]] == 301 }, {x, -1, 1}, {y, -1, 1}]

Obtained in Wolfram Mathematica 9

We see, that these curves may intersect only for $t = \pm 1$, hence the condition (38) may fail only for one k. So, in this case the defect numbers of the problem (36), (4) may only be equal to zero and one. Therefore, we can suppose that if the defect numbers of the problem (1), (4) for fourth order equation (1) in elliptic case are finite, they are not greater than one for an arbitrary location of the characteristic equation's roots, but this supposition must be proved.

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A unified approach to the data fitting problem

Gevorgyan L.
National Polytechnic University of Armenia
E-mail: levgev@hotmail.com

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Abstract : From Wikipedia, the free encyclopedia
"Curve fitting is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constraints. Curve fitting can involve either interpolation, where an exact fit to the data is required, or smoothing, in which a "smooth" function is constructed that approximately fits the data. Fitted curves can be used as an aid for data visualization, to infer values of a function where no data are available, and to summarize the relationships among two or more variables."
We analyze some interpolation and approximation methods and propose a general method of data fitting procedure. Two computer programs are presented.

The oldest data fitting algorithm is described by the Lagrange interpolation formula. For the given data set $M=\left\{\left(x_k, y_k\right)\right\}_{k=0}^{n}$, where no two x_k (the nodes or knots) are the same, it is necessary to find the polynomial of the least degree that at each x_k assumes the corresponding value y_k. This problem is always solvable and the solution is unique. Indeed, if the interpolating polynomial has the form
$P(x)=a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1 x+a_0$,
then

$$\begin{cases} a_n x_0^n+a_{n-1}x_0^{n-1}+\cdots+a_1 x_0+a_0=y_0, \\ a_n x_1^n+a_{n-1}x_1^{n-1}+\cdots+a_1 x_1+a_0=y_1, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n x_n^n+a_{n-1}x_n^{n-1}+\cdots+a_1 x_n+a_0=y_n. \end{cases}$$

The principal matrix of this system is the Vandermonde matrix V with non-zero determinant, hence the solution (interpolating polynomial) exists and is unique. The matrix V is notoriously bad conditioned so usually another way is chosen. One starts by the product $\omega(x)=\prod_{k=0}^{n}\left(x-x_k\right)$
and introduce the Lagrange fundamental polynomials

$$l_k(x)=\frac{\omega(x)}{\left(x-x_k\right)\omega'\left(x_k\right)}, \qquad (1)$$

satisfying the biorthogonality condition $l_k(x_m) = \delta_{km}, k, m = 0, 1, \cdots, n.$

Finally $P(x) = \sum_{k=0}^{n} y_k l_k(x).$

The uniqueness of interpolation problem's solution implies that

$$x^m \equiv \sum_{k=0}^{n} x_k^m l_k(x),\ m = 0, 1, \cdots, n-1,$$

Conceived to serve as a tool in the investigation of functions, interpolation polynomials suffer two serious flaws. The first is the polynomial wiggle, i.e. increasing the degree of

Fig.1. Lagrange fundamental polynomials

the polynomial makes the oscillations very large (Runge phenomenon, [5]). The second is the impossibility to handle arbitrary plane curves (potentially multi-valued functions).

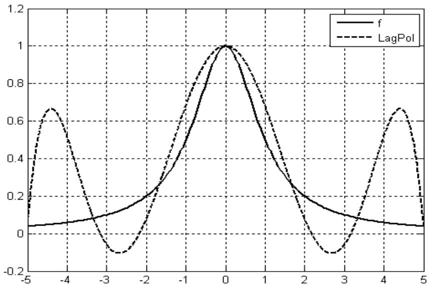

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Fig.2. Function $f(t) = \dfrac{1}{1+t^2}$ and the interpolating polynomial

For the same knots another interpolation formula may be obtained, introducing Hermite-Fejer basic polynomials

$$h_k(x)=\left(1-\frac{\omega''(x_k)}{\omega'(x_k)}(x-x_k)\right)l_k^2(x).$$ One has $\sum_{k=0}^{n} h_k(x)\equiv 1$.

Example. For the interval $[-1;1]$ and two nodes $\{-1;1\}$ the Hermite-Fejer basic polynomials are $h_1(x)=\frac{x^3-3x+2}{4}, h_2(x)=\frac{2+3x-x^3}{4}$ (see Fig. 3) and the interpolating polynomial for x is $\frac{3x-x^3}{2}$.

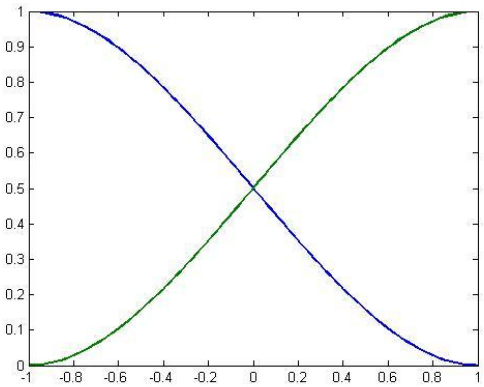

Fig. 3. Hermite-Fejer basic polynomials

According to Fejer's theorem [3]}, ch.4, § 7) the Hermite-Fejer interpolating polynomials, constructed by the nodes, consisting of zeros of the Chebyshev polynomials of the first kind tend to any continuous on $[-1;1]$ function f.

The polynomial wiggle shortcoming is remedied by the introduction of the interpolation by splines.

To this end usually are used cubic polynomials, different for each pair of neighboring nodes, regularized such that at each node the resulting function S and its derivatives up to the order two are continuous. Each cubic polynomial is determined by 4 constants, the number of intervals is n, so the problem totally concerns $4n$ parameters. At each inner node 4 conditions are imposed and 2 additional conditions are imposed at the first and the last nodes, so we get $4n-2$ conditions. Two constants remain free and may be chosen arbitrary, depending on the nature of the spline ("natural- $S''(x_0)=S''(x_n)=0$ ", "clamped-

$S'(x_0) = f'(x_0), S'(x_n) = f'(x_n)$ ", "not-a-knot"-
$S'''(x_1 - 0) = S'''(x_1 + 0), S'''(x_{n-1} - 0) = S'''(x_{n-1} + 0)$ splines). All these conditions lead to a diagonally dominated tridiagonal system of linear equations, which is uniquely solvable. In practice the spline is found as the solution of that system. We need more complicated and indirect construction. Further we will consider only natural splines.

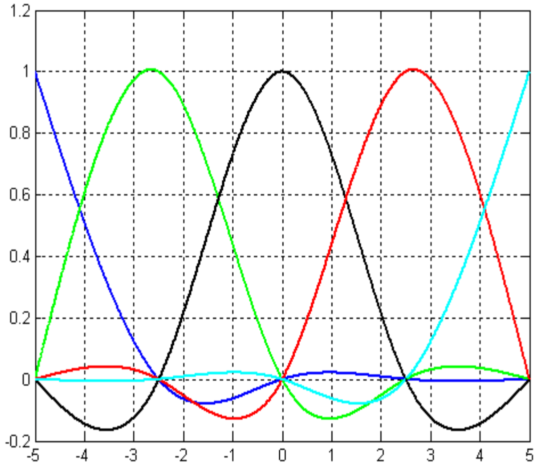

Fig. 4. Basis splines

First we find so called basis splines $\{s_k\}$ that are the solutions of the problem $s_k(x_m) = \delta_{km}$ and afterwards the spline is reconstructed by the equality

$$S(x) = \sum_{k=0}^{n} y_k s_k(x).$$

Evidently $\sum_{k=1}^{n} x_k^m s_k(x) \equiv x^m, m = 0,1.$

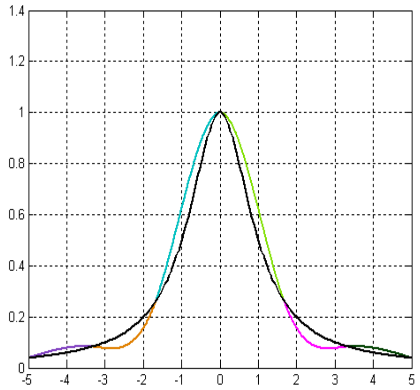

Fig.5. Interpolation by a spline of $f(t) = \dfrac{1}{1+t^2}$.

The smoothing idea is implemented in the Bernstein polynomials. The general case of arbitrary interval $[a;b]$ is reduced to $[0;1]$ and set of weights (Bernstein basis polynomials) $b_k(t) = C_n^k t^k (1-t)^{n-k}, t \in [0;1]$ are introduced. Note that the maximum of b_k is attained at the point $\dfrac{k}{n}$, which will serve as equidistant knots. For any continuous on $[0;1]$ function f the sequence of Bernstein polynomials

$$B_n(f,t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_k(t)$$

converges uniformly on $[0;1]$ to f. Note that

$$0 \le b_k(t) \le C_n^k \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} < 1, k = 1, 2, \cdots, n-1.$$

One has ([2], 6.2.4-6.2.6)

$$\sum_{k=0}^{n} b_k(t) \equiv 1, \qquad (2)$$

$$\frac{1}{n}\sum_{k=0}^{n} k b_k(t) = t, \qquad (3)$$

$$\frac{1}{n^2}\sum_{k=0}^{n} k^2 b_k(t) = \frac{t}{n} + \frac{n-1}{n} t^2. \qquad (4)$$

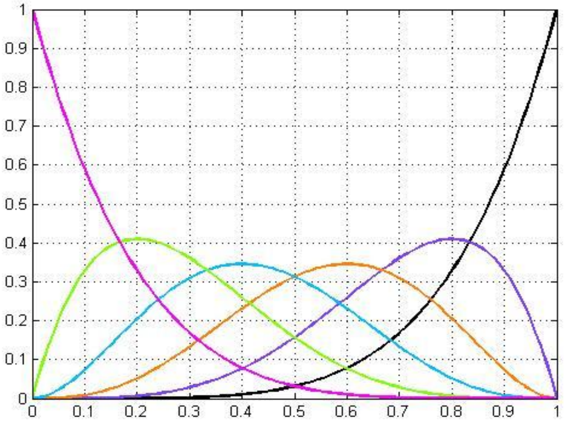

Fig. 6. Bernstein basis polynomials.

Bohman-Korovkin Theorem **[3]**. Let $L_n\left(n \geq 1\right)$ be a sequence of positive linear operators acting in the space of continuous functions $C\left(\left[a,b\right]\right)$. If $\left\|L_n f - f\right\|_\infty \to 0$ for $f\left(x\right)=1, x$ and x^2, then $\left\|L_n f - f\right\|_\infty \to 0$ for all $f \in C\left(\left[a,b\right]\right)$.

Formula (4) shows that the approximation by the Bernstein polynomials can not be faster than $1/n$. Real situation is even worse. In fact, one can show that in order to have a maximum error smaller than 0.01 one needs at least a degree of $1.6 \cdot 10^7$.

Let $\left[a;b\right]$ be any segment of the real axis and $\left\{e_k\left(t\right)\right\}_{k=0}^n$ be a set of continuous linearly independent functions defined on $\left[a;b\right]$.

Definition 1. The set $\left\{e_k\left(t\right)\right\}_{k=0}^n$ forms a partition of unity if $\sum_{k=0}^n e_k\left(t\right) \equiv 1$.

Remark. Usually in the definition of the partition of unity $\left\{e_k\left(t\right)\right\}_{k=0}^n$ the following condition $0 \leq e_k\left(t\right) \leq 1, \forall t \in \left[0;1\right]$ is imposed.

Actual definition does not exclude the possibility that some functions may admit negative values also, as well values, greater than 1 may occur.

Considered above Lagrange fundamental polynomials, Hermit-Fejer basis polynomials, basis splines and the Bezier basis polynomials form a partition of unity.

Definition 2. Let $\{t_k\}_{k=0}^{n} \subset [a;b]$ be a set of pairwise different points and $\{e_k(t)\}_{k=0}^{n}$ - set of functions, defined on $[a;b]$. We say that they are biorthogonal if

$$e_k(t_m) = \delta_{km}, k, m = 0, 1, \cdots, n. \qquad (5)$$

All mentioned above partitions of unity, except the Bezier basis polynomials are biorthogonal with corresponding set of knots.

Note that the biorthogonality condition (5) forces the fitting curve to pass by the interpolation nodes, i.e. in this case one gets the interpolation polynomial.

Definition 3. We say that the set $\{e_k(t)\}_{k=0}^{n}$ reproduces the function f if

$$f(t) \equiv \sum_{k=0}^{n} f(t_k) e_k(t).$$

The Lagrange interpolation formula reproduces monomials $\{t^k\}_0^{n-1}$. Formulas (2)-(4) mean that the approximation by the Bernstein polynomials as well (natural) splines reproduce the functions 1 , t and do not reproduce t^2. The Hermite-Fejer basic polynomials reproduce the constants.

In order to keep data fitting curve in the "narrowest" possible vicinity of data set M, it is natural to seek a partition (if this is possible) such that the corresponding Lagrange fundamental polynomials have "smallest" collective deviation from $Ox-$ axis. More precisely, consider the set of points $\{x_k\}$ and corresponding Lagrange fundamental polynomials $\{l_k(t)\}_{k=1}^{n}$. Denote by $K : C(a;b) \to C(a;b)$ the projection operator putting in correspondence to any function f the polynomial $(Kf)(x) = \sum_{k=1}^{n} f(x_k) l_k(x)$. Denote the Lebesgue constants (the norm of the operator K) by

$$\Lambda_n = \sup_{x\in[a;b]} \sum_{k=1}^{n} |l_k(x)|.$$

Let $\mathcal{P}_n$ be the set of all polynomials with the degree not exceeding n and

$$e_n(f) = \inf_{p\in\mathcal{P}_n} \|f - p\|.$$

It is known that $\|f - Kf\| \le (1 + \Lambda_n) e_n(f)$.

For any choice of points $\{x_k\}$ one has $\Lambda_n \geq \frac{\ln n}{22}$.

The "almost" optimal value of Λ_n is attained (for the segment $[-1;1]$) when the nodes coincide with the "expanded Chebyshev nodes" (roots of the Chebyshev polynomials of the first kind, scaled such that $x_1 = -1$ and $x_n = 1$). The norm of this operator is equal [1] to $\Lambda_n = \frac{2}{\pi}\ln n + 1 - \theta_n, 0 \leq \theta_n < \frac{1}{4}$.

More versatile tool, permitting handling multi-valued functions and treating arbitrary knots is supplied [4] by the Bezier construction. The Bezier curves are defined by parametric equation as the convex linear combination of the points M

$$B(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{k=0}^{n} \begin{pmatrix} x_k \\ y_k \end{pmatrix} b_k(t),$$

hence the Bezier curve lies in the convex hull of M. As only w_0 at $t = 0$ and w_n at $t = 1$ are equal to 1 the Bezier curve, in contrast with the Lagrange polynomial, passes by the first and the last nodes $(x_0; y_0)$ and $(x_n; y_n)$.

We propose the following general formula. The fitting curve is defined by parametric equation

$$\begin{cases} x(t) = \sum_{k=0}^{n} x_k e_k(t), \\ y(t) = \sum_{k=0}^{n} y_k e_k(t). \end{cases} \quad (6)$$

Note that if the set $\{e_k(t)\}_{k=0}^{n}$ reproduces the linear function then formula (6) includes as particular case interpolation formulas. The Bezier curves correspond to the choice of Bernstein basis polynomials as the set $\{e_k(t)\}_{k=0}^{n}$.

Example. Let $a = -1, b = 1, t_k = -\cos\frac{\pi(2k-1)}{2n}, k = 1, \cdots, n.$

The Lagrange fundamental polynomials defined by formula (1) are

$l_k(t) = (-1)^k \frac{T_n(t)}{n(t - t_k)} \sin t_k$, where T is the Chebyshev polynomials of the first kind.

Below the MatLab code of corresponding plot is appended.

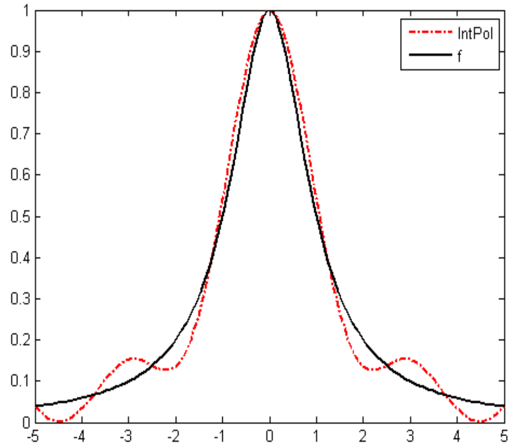

Fig.7. Function $f\left(t\right)=\frac{1}{1+t^2}$ and the interpolating polynomial

```
function ABInter(a,b)
n=length(a);
p=linspace(pi/2/n,pi-pi/2/n,n);
m=max(-cos(p));
r=-cos(p)/m;
syms t
for k=1:n
   f=prod(t-r(1:k-1))*prod(t-r(k+1:n));
   g=prod(r(k)-r(1:k-1))*prod(r(k)-r(k+1:n));
   l(k)=f/g;
end
L=dot(a,l);
M=dot(b,l);
s=linspace(-1,1);
x=subs(L,t,s);
IntPolCh=subs(M,t,s);
plot(x,IntPolCh,'-.r','Linewidth',1.5)
```

The next plot shows the difference between the Bezier curve and the fitting curve constructed by the formula (6). Input data is generated by mouse clicks.

```
function Lintsym(n)
set(axes,'Xlim',[0,1],'YLim',[0,1])
g_fig=gcf;
set(g_fig,'Position',[10 10,700 680]);
hold on
```

```
h=text(.35,1.05,'Choose 1-st control point','Fontsize',14);
a=zeros(1,n);b=zeros(1,n);
for k=1:n
[a(k) b(k)]=ginput(1);
plot(a(k),b(k),'Color','k','Marker','*')
delete(h)
  if k==1
     h=text(.35,1.05,'Choose 2-nd control point','Fontsize',14);
  elseif      k==2
     h=text(.35,1.05,'Choose 3-rd control point','Fontsize',14);
  elseif k<n
     h=text(.35,1.05,['Choose ',num2str(k+1),'-th control point'],'Fontsize',14);
  else
     h=text(.2,1.05,'Interpolation curve and the control polygon','Fontsize',14);
  end
end
plot(a,b,'k')
hold on
syms t
n=length(a);
p=linspace(pi/2/n,pi-pi/2/n,n);
m=max(-cos(p));
r=-cos(p)/m;
syms t
for k=1:n
   f=prod(t-r(1:k-1))*prod(t-r(k+1:n));
   g=prod(r(k)-r(1:k-1))*prod(r(k)-r(k+1:n));
   l(k)=f/g;
end
L=dot(a,l);
M=dot(b,l);
s=linspace(-1,1);
x=subs(L,t,s);
FitC=subs(M,t,s);
plot(x,FitC,'-.r','Linewidth',1.5)
 for k=1:n
   c(k)=nchoosek(n-1,k-1)*t^(k-1)*(1-t)^(n-k);
end
u=c*a';
v=c*b';
x=subs(u,t,linspace(0,1));
Bez=subs(v,t,linspace(0,1));
plot(x,Bez,'g--','LineWidth',2)
axis 'equal'
hold on
```

```
plot(a,b)
legend('FitC','Bez')
```

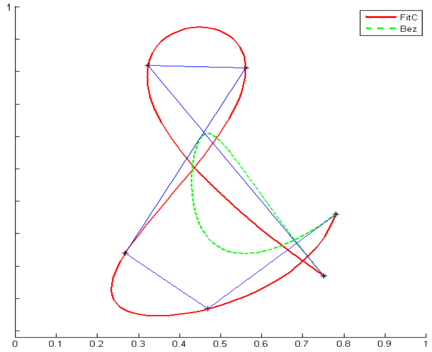

Fig.8. Interpolation curve, Bezier curve and the control polygon

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Introduction to phantom graph

Philip LLOYD, New Zealand

philiplloyd1@gmail.com

AMS 2010 mathematical subject classification: 97I80
KEY WORDS: Complex Analysis

Abstract : While teaching "Solutions of Quadratics" I was emphasising the idea that, in general, the solutions of equations such as $ax^2 + bx + c = 0$ are obviously the points where the graph of $y = ax^2 + bx + c$ crosses the x axis. I started to be troubled by the special cases of parabolae that do not even cross the x axis. We say that these equations have "complex solutions" but physically, where are these solutions? With a little bit of lateral thinking, I realised that we can physically find the actual positions of the complex solutions of any polynomial equation and indeed many other common functions! The theory also shows clearly and pictorially, why the complex solutions of equations with real coefficients occur in conjugate pairs.

INTRODUCTION

This is the basic graph of $y = x^2$ and if we only use real values of x we only obtain positive values of y. Fig 1

$x = \pm 1$ we get $y = 1$
$x = \pm 2$ we get $y = 4$
$x = \pm 3$ we get $y = 9$

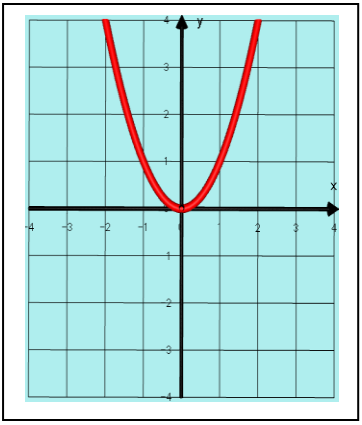


Fig 1

However, if we allow values of x such as:

$x = \pm i$ we get $y = -1$
$x = \pm 2i$ we get $y = -4$
$x = \pm 3i$ we get $y = -9$

The insight, is to allow a complex x PLANE but with just a real y AXIS.

This produces a sort of "*phantom*" parabola underneath the basic parabola and at right angles to it.

I have discovered that nearly ALL curves have these extra "*phantom*" parts and more importantly, this has an intriguing connection with the Fundamental Theorem of Algebra.

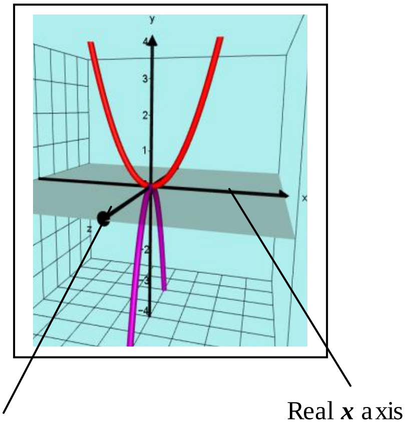


Fig 2

"PHANTOM GRAPHS".

Basically, the Fundamental Theorem of Algebra states that polynomial equations of the form: $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ px^2 + qx + r = 0$ will have n solutions. (where n is a positive integer)

This is often interpreted as:
"*The solutions of an equation $f(x) = 0$ are where the graph of $y = f(x)$ crosses the x axis*"
but this only finds the solutions which are REAL numbers.

Consider the equation $x^2 - 4x + 3 = 0$
The graph of $y = x^2 - 4x + 3$ is as shown in Fig 3

The graph crosses the x axis at $x = 1$ and $x = 3$ so the solutions are $x = 1$ and 3

In this case, the phantom hanging below had no part to play in this logic.

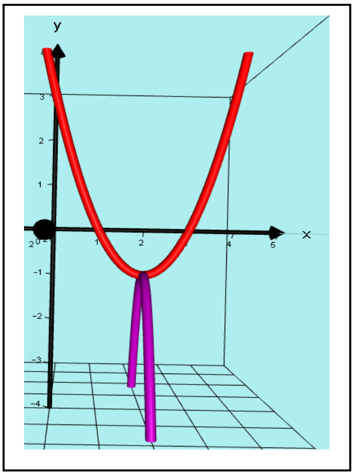


Fig 3

However, consider the equation $x^2 - 4x + 4 = 0$

The graph of $y = x^2 - 4x + 4$ is as shown in Fig 4.

In this case, the top half of the parabola crosses the x axis at $x = 2$ AND the bottom half of the parabola (*the phantom*) also crosses the x axis at $x = 2$. *(a double solution)*

The graph goes through the point (2, 0) twice.

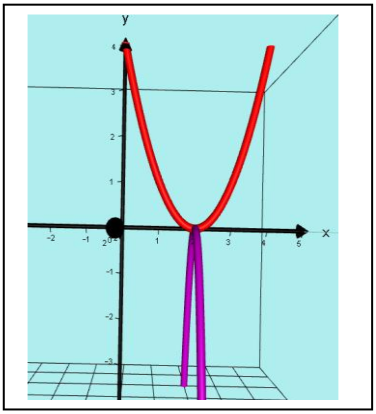


Fig 4

Of course, the most interesting case is when the basic top half of the parabola would not normally cross the *x AXIS* at all but its *phantom* would cross the *complex x PLANE*!
Consider the equation $x^2 - 4x + 5 = 0$
The graph of $y = x^2 - 4x + 5$ is as shown in Fig 5.

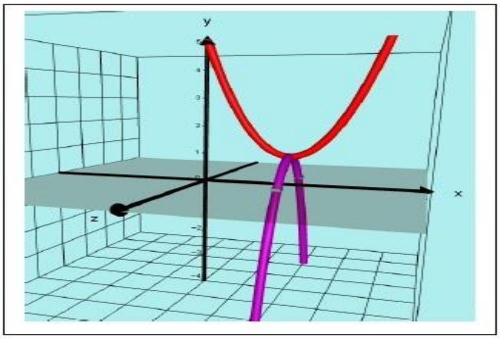


Fig 5

The *phantom* crosses the *x plane* at $x = 2 + i$ *and* $x = 2 - i$ as shown in Fig 5 and these are the complex solutions of the equation.

We can now re-state the Fundamental Theorem of Algebra as:
"The solutions of an equation $f(x) = 0$, *whether they are real or complex,* are where the graph of $y = f(x)$ crosses the *COMPLEX x PLANE*".

Clearly, we can see that any parabola of the form: $y = ax^2 + bx + c$ (with its phantom) will cross any horizontal plane (which represents any real y value) exactly two times. Fig 6.

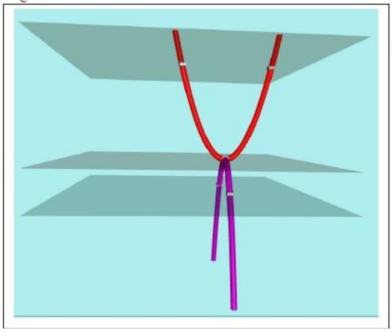


Fig .6

In my website www.phantomgraphs.weebly.com you will find detailed working to show HOW CUBIC functions of the form: $y = ax^3 + bx^2 + cx + d$ each have 2 phantoms emanating from their maximum and minimum points. See Fig 7.

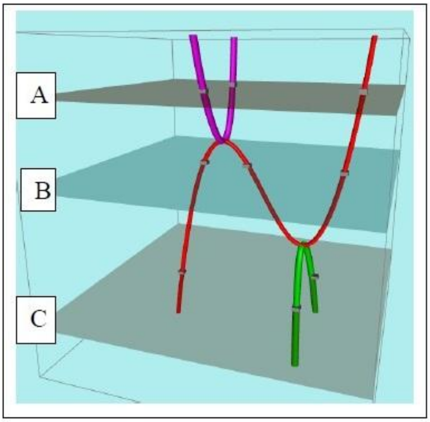


Fig.7

We know that any Cubic equation of the form: $ax^3 + bx^2 + cx + d = 0$ will have 3 solutions.Sometimes we have 3 REAL solutions as for the intersections with the middle Plane B in Fig 7 and sometimes we have 1 real solution and 2 complex solutions as on Planes A and C. Fig 7.

This is a typical QUARTIC graph. Fig 8 showing 1 maximum and 2 minimum points.

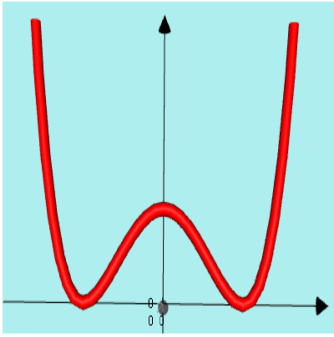


Fig 8

This is the same Quartic graph with its 3 *phantoms* emanating from each turning point. Fig 9.

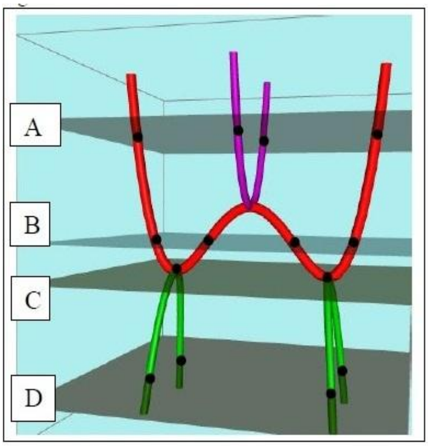


Fig.9

The intersection points with the 4 planes A, B, C and D with the graph, are marked. Plane A shows 2 real solutions on the basic RED curve and 2 imaginary solutions on the PURPLE *phantom*.

Plane B shows 4 real solutions on the basic RED curve.

Plane C shows 2 double real solutions which lie on the basic red curve and the GREEN *phantoms*.

Plane D shows 4 imaginary solutions on the two GREEN *phantoms*. (2 sets of conjugate roots)

Clearly a QUARTIC curve will pass through ANY horizontal plane 4 times.

In my website www.phantomgraphs.weebly.com I have found that all sorts of graphs, not just polynomials, have some amazing and surprising *PHANTOMS*.

Examples are $y = \frac{2x}{x^2 - 1}$ **,** $y = \cos(x)$**,** $y = e^x$**,** $y^2 = x(x-3)^2$ **and many more.**

One particularly lovely surprise was the hyperbola $y^2 = x^2 + 25$

Fig 10

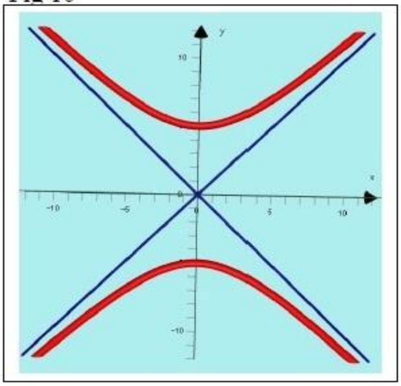


Fig 10

There are clearly NO real y values in the interval $-5 \le y \le 5$ .So I decided to calculate complex x values for y values such as $y = 3$ *substituting in* $x^2 + 25 = y^2$ *we obtain*

$$x^2 + 25 = 9$$
$$x^2 = -16$$
$$\text{so} \quad x = \pm 4i$$

Similarly, if $y = 4$, $x = \pm 3i$
and if $y = 0$, $x = \pm 5i$

These of course are points on a CIRCLE of radius 5 units and this *phantom circle* joins the two halves of the hyperbola! See Fig 11.

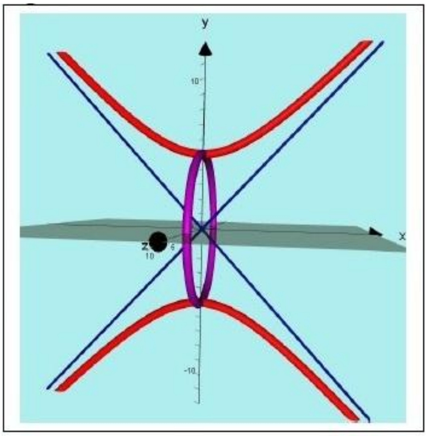


Fig 11

When I first worked on *phantom graphs* I used to calculate the complex points as above and then I made Perspex models to demonstrate the graphs clearly in 3 dimensions. In order to draw the graphs in *Autograph* I had to work out the actual *equations* of the *phantoms*.

I will demonstrate the method for the hyperbola shown above.

Firstly we have to allow *complex x* values so that
when x values appear in the equation we need to replace them with $(x + iz)$

The above equation becomes $y^2 = (x + iz)^2 + 25$ *Equation 1*

Expanding and rearranging: $y^2 = (x^2 - z^2 + 25) + (2xz)i$

The important idea now is that phantom graphs can have *complex x values* BUT the *y values* must only be REAL numbers.

This means the imaginary part of y must be zero.
That is $2xz = 0$ *so* $x = 0$ *or* $z = 0$
If $z = 0$ then ***Equation 1*** simply becomes $y^2 = x^2 + 25$ which is the original hyperbola.
If $x = 0$ then ***Equation 1 becomes $y^2 = (iz)^2 + 25$ that is $y^2 = -z^2 + 25$ or in its more familiar form $y^2 + z^2 = 25$ which is the phantom circle joining the two halves of the hyperbola!***

By far the most challenging complex algebra was needed in finding the equations of the *phantoms* for the function $y = \frac{x^4}{x^2 - 1}$ see Fig 12

Fig 12

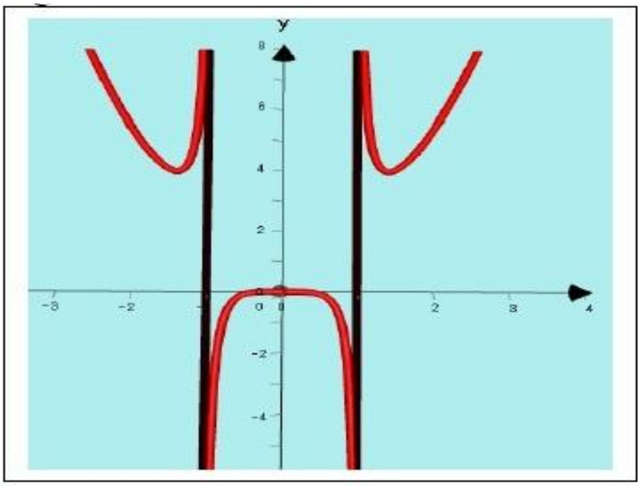


Fig 12

We can see that there are no real values of ***y*** in the interval $0 \le y \le 4$

Interestingly, if we consider a general ***y*** value such as $y = c$ we get $\dfrac{x^4}{x^2 - 1} = c$

which produces a typical quadratic equation $x^4 - cx^2 + c = 0$ which of course has 4 solutions.

If we draw $y = 5$ on **Fig 12** it crosses 4 times but if we draw $y = -2$ it only crosses twice.

But when we consider the graph with its Phantoms, we see that **any horizontal plane** $y = c$ crosses the graph 4 times which further verifies the truth of the Fundamental Theorem of Algebra.

Incidentally, the equation of the bottom purple phantom is: $y = 1 - z^2 - \dfrac{1}{z^2 + 1}$

and the equation of the top blue phantom is: $y = x^2 - z^2 + 1 + \dfrac{(x^2 - z^2 - 1)}{(x^2 - z^2 - 1)^2 + 4x^2z^2}$

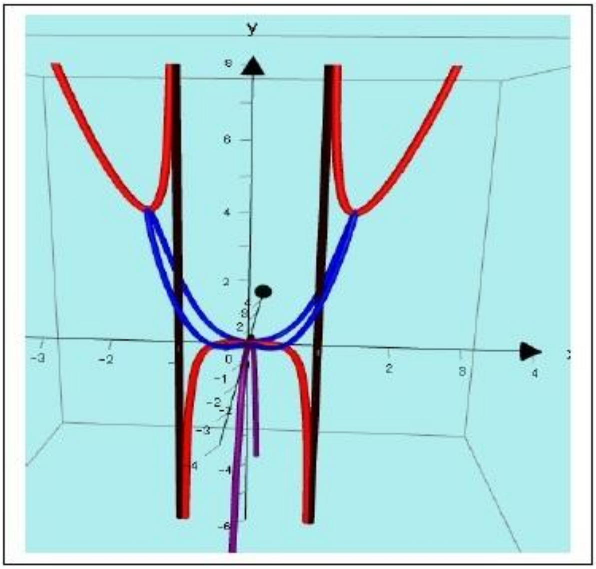


Fig 13

<u>The Intriguing Function $y = x^x$</u>

When we plot this graph using Autograph
we obtain the red graph in **Fig 14.**

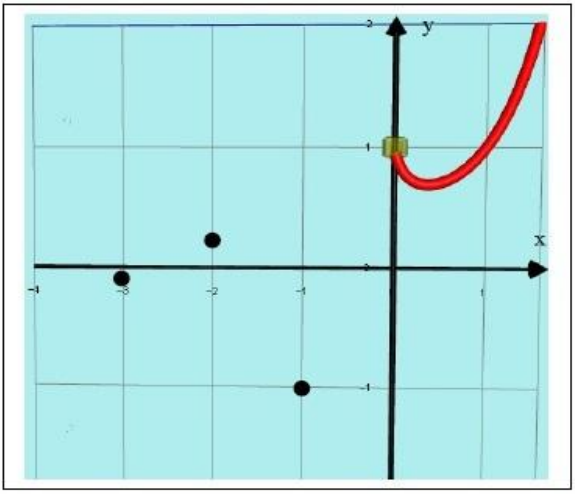


Fig 14

If $x = -1$ we can work out $y = (-1)^{-1} = -1$ so we can also plot the point (– 1, – 1)

Similarly if $x = -2$ then $y = (-2)^{-2} = +0.25$ and if $x = -3$ then $y = (-3)^{-3} = -0.037$

These isolated points seemed strange so I decided to find more points using a graphics calculator and I realised that complex numbers appear!
For example: If $x = -0.3, y = (-0.3)^{-0.3} = 0.84 - 1.16i$
If $x = -0.9, y = (-0.9)^{-0.9} = -1.05 - 0.34i$
If $x = -1.5, y = (-1.5)^{-1.5} = 0 + 0.54i$
This graph is different from the types of phantom graphs previously considered because they have only **REAL *y* values** but **complex *x* values** are allowed.
That is, a ***real y axis*** but a ***complex x plane***. In this case, we have only **REAL *x* values** but **complex *y* values** are produced. That is, a ***real x axis*** but a ***complex y plane***. I calculated a lot such points and the very surprising result was this delightful SPIRAL! **Fig 15**

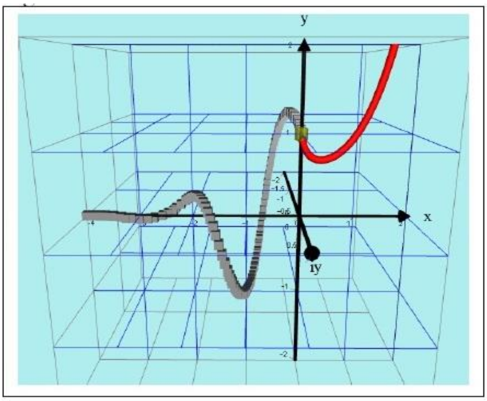
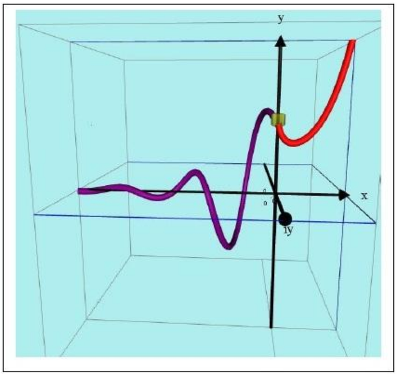
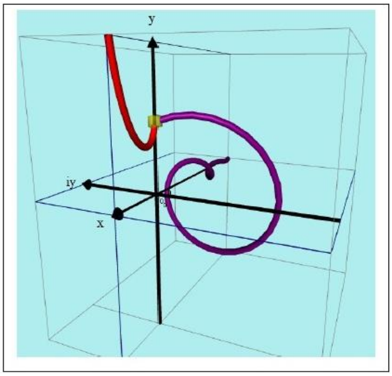


Fig 15

The curve spirals around the ***x*** axis.

I will finish off with views of the **SPIRAL** from two different angles.





CONCLUSION : It is an interesting concept that graphs, even as simple as $y = x^2$, which were previously assumed to exist only in the "*x, y plane*" can be shown to have a 3 dimensional aspect if we include only those complex *x* values which still produce REAL y values.This concept makes The Fundamental Theorem of Algebra far more meaningful in the form: "The solutions of an equation *f(x) = 0, whether they are real or complex,* are where the graph of *y = f(x)* crosses the *COMPLEX x PLANE*".

Note: All the graphs have REAL *y* values and we only use the complex *x* values which produce real *y* values. If any complex y values were considered as well as complex *x* values, the graphs would need 4 dimensions.

REFERENCES :
These ideas in this paper are completely my own original work.
I encourage readers to see a full representation of my work in my website:
www.phantomgraphs.weebly,com
Originally, I made many Perspex models of my phantom graphs as shown below:

However, I would like to acknowledge Douglas Butler and Simon Woodhead from the AUTOGRAPH organisation for their encouragement and support in transferring my models into graphs in the Autograph system.

Philip Lloyd
philiplloyd1@gmail.com
New Zealand

A geometrical Isomorphism and the relativity of geometry

George Mpantes , Greece
mpantes_2@yahoo.gr

AMS 2010 mathematical subject classification : 05C60 , 40E99
Key words : Isomorphism, Inversion Euclid's axioms .

Abstract : The one –to-one correspondence of the geometrical *inversion*, inverts the Euclidean plane to Ideal plane which contains all the points of the first except one (the center of inversion) plus one point at infinity. The inverse of Euclidean straight lines in Ideal plane, is a system of circles passing from the center of inversion, which complete the axioms of Euclid. So these circles are the Ideal straight lines of the Ideal plane. Now the inversion becomes an *isomorphism* and the geometry of the two spaces are identical except for superficial differences in terminology and notation.

Contents

1. The isomorphism
2. Inversion : the transformation of the plane to itself
3. The axioms of Euclid's straight line
4. The inversion as an isomorphism
5. Ideal geometry is imaginary but mathematically consistent
6. Comment

1. The isomorphism

In mathematics, we study the properties of postulate sets (independence, completeness, categoricalness, complete independence etc). That was first brought into prominence by Hilbert's "*Grundlagen der Geometry*". There, we meet the concept of **isomorphism** in it's general expression:
A postulate set P is said to be categorical if any two interpretations of P are *isomorphic.*

Two interpretations I and I' of a postulate set P are isomorphic if one can set up a one-to-one correspondence between the elements of I and those of I' in such a way as "to be preserved by the relations and the operation of P. It follows that if two interpretations I and I' of a postulate set are isomorphic , then any true (false) proposition p in interpretation I becomes a true (false) proposition p' in interpretation I' when we replace the elements e, the relations r and the operations

o on the elements in p, by their corresponding e´ ,r´, and o´. (H. Eves) Two isomorphic interpretations of a postulate set P are , except for superficial differences in terminology and notation, identical; two isomorphic groups G and G' (they are the interpretations I and I' of the postulate set of groups) cannot be distinguished from the view of the theory of groups. An isomorphism can be looked as a renaming of the elements of G in elements of G' .

Anecdote: a mathematician was asked if he believes in God. Answer: Yes, via an isomorphism

Removing our interest in the area of geometry, then all the above are translated in what is known from the theory of surfaces: two surfaces E_1 and E_2 are called **isomorphic** if it is possible to define a one-to-one correspondence of all the points of E_1, on the points of E_2 so that each "straight line" of E_1 corresponds in a "straight line " on E_2. Then the geometries of the two surfaces are identical: each proposition in one (geometry E_1) applies to another (the geometry of E_2). In this result, there are the bases of the Euclidean models of the non-Euclidean geometries, that the Ideal geometry of this article, is the first trial.

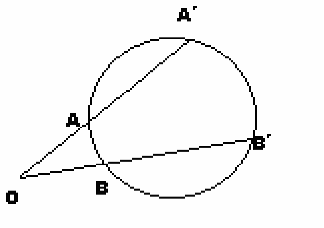
In the sequel, we shall set up a one-to-one correspondence between the points of the Euclidean plane into itself, proving that this correspondence is an isomorphism.

2. Inversion : the transformation of the plane to itself .

Let Π a fixed circle of center O and radius r , and let P be any point in the plane of Π. Then the point P' on the ray OP such that $OP.OP'=r^2$ is called the **inverse** of P with respect to circle Π. We add to the plane a single ideal point at infinity. If P≡0, then P' is taken as this ideal point. Circle Πis called the circle of inversion , point O the centre of inversion , and r^2 the power of inversion. There is set up a one-to-one correspondence between the points of the plane of Π; to every point there is a corresponding point , the points of the curve C will invert into the points of a curve C' , called the inverse of C. We can prove the following **theorems** concerning this transformation of inversion

Th.1. if P' is the inverse of P , then P is the inverse of P'

Th. 2. A point inside the circle of inversion inverts into a point outside the circle of inversion; a point outside the circle of inversion inverts into a point inside of the circle of inversion; a point on the circle of inversion inverts into itself



σχ. 1

Th.3.the necessary and sufficient condition that two shapes are inverse, is that any two pairs of corresponding points not collinear, are concyclic.

Consider the points A, B and the inverse A´ , B' for inversion (O, r^2)[1]. (Fig. 1) is then OA.OA = OB.OB = r^2 so the four points as long as they are not collinear, are con-cyclic. Conversely, it is easily demonstrated that two shapes, between which there is a correspondence such that any two pairs of corresponding points to be con-cyclic, then the shapes are homologous to an inversion.

Th.4. a straight line through the center of inversion inverts into itself
Th.5 a circle orthogonal to the circle of inversion inverts into itself
Th.6. A straight line that does not pass through the center of inversion, inverts into a circle that does not pass through the center of inversion.
Th. 7. the inverse of a circle that does not pass through the center of inversion, is a circle that does not pass through the circle of inversion, and homothetic to this.
Th.8. any circle through a pair of inverse points P and P' cuts the circle of inversion orthogonally.
Th.9. The inverse of a circle that pass from the center of inversion O, is a straight line parallel to the tangent of the circle at O. (fig.5)

Th. 10. Two intersecting circles C′ and C′′ orthogonal to the same circle C, are intersecting at points P and P' which are inverse relative to the circle C. (Fig. 2)

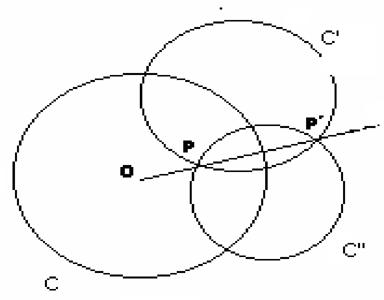


fig. 2

Th.11. a given circle may be inverted into itself by the use of any given exterior point as center of inversion.
Th.12. in an inversion, the angle between two intersecting curves is equal to the corresponding angle between the two inverse curves. A transformation that preserves angles between curves is called **conformal** transformation. So, inversion is a conformal transformation.

Th.13. the distance of the inverse points.
Without harming the generality, we consider the radius of inversion r = 1
In Fig. 3

is OA.OA′ = OB.OB′ = 1. The triangles OAB, OA′B′ are similar.

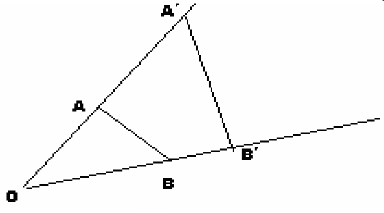


fig. 3

therefore
A′B′ / AB = OA′ / OB ⇒

[1] The circle (O,r) is not in the figure.

A´B´=AB.OA´/OB=AB.OA´OA/OB.OA=AB/OA.OB
finally
A´B´=AB/OA.OB ……….(1)

If points A, B, A', B' are collinear the same formula applies to the distance of A', B'
More details and additional properties for the inversion we can find in the γεωμετρία Κανέλλου and in the geometry of the Jesuits.

The system of circles passing through a fixed Point..

We shall examine now the representation of ordinary plane geometry by the geometry of a system of circles through a fixed point O, with the results of the above transformation of the plane on itself (the inversion with center O and radius 1). It is convenient to speak for the plane of the straight lines and the plane of circles , as two separate planes (the second as **Ideal plane**). We have seen that to every straight line in the plane of the straight lines , there corresponds a circle in the plane of circles. We shall call these circles ***Ideal* straight *lines.*** The ***Ideal points*** will be the same as ordinary points , except that the point O will be excluded from the domain of the Ideal points, plus a point at infinity. As ***angle of ideal lines***, we define the angle of the archetype straight lines through the inversion, as inversion preserves angles between two inverse curves.

If we prove that the correspondence of inversion is an isomorphism of the Euclidean plane to the Ideal plane, then the geometries of the two planes will be identical. The properties of the set of circles could be established from the knowledge of the geometry of the straight lines, and every proposition concerning points and straight lines in the one geometry could at once be interpreted as a proposition concerning points and circles in the other.

3. The axioms of Euclid's straight line .

The axioms of Euclid are:

1. There is exactly one straight line through two distinct points.

2. Every straight line can be extended indefinitely and is open from both ends, has infinite length. For any two points A, B there is always another C to B is "between"

A and C. The meaning of " between-ness" , is basic for Euclidean geometry.

3. You can draw a circle with any center and radius. This axiom seems to be unrelated to the points and straight lines. But beware if the Euclidean definition of the circle "which is the line where all points equidistant from one another," we will see that this axiom ensures something for distance. How do we know that the distance of the ends of diabetes remains the same when rotated; It ensures that the "distance" in the plane (space), as if set, should ensure the unchanged length for a segment that is moved from one place to another.

4. All right angles are equal. Again we need to know the proper Euclidean definition to interpret the axiom: "When two intersecting lines form the successive angles equal then each of them is a right angle. So the fourth axiom is equivalent to the case that the straight lines have not zig-zag, "break", angular points. Let us remember the greatest circle of the sphere .

5. The most famous axiom in the history of science: from a point outside a straight one parallel is conducted to this. Parallel lines are in the same level that as if extrapolated do not intersect.

The question that comes here is: is Euclidean straight of our daily experience, the only line that satisfies the above axioms? though the world around us infuses us with a strong intuition about what is straight line, can the axiomatic construction of Euclid be applied to another line after appropriate axiomatic assumptions about space? If so, we will begin to falter in the absolute of Euclidean space. This thought, ie the search of another line that the axioms of Euclid are valid, and the investigations of the consequences of this geometric phenomenon leads us to study the isomorphism mentioned above. This isomorphism is a first example of understanding the relativity of geometry and is reported in the book of Bonola, "Non Euclidean Geometry".

4. The inversion as an isomorphism .

The first axiom

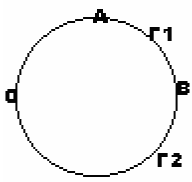


fig.4

Any two different Ideal points A,B determine the Ideal line A, B (fig.4), just as in Euclidean geometry, as three points (O,A,B) define a unique circle. So the *first axiom of Euclid* is valid in Ideal plane.

Definition of ideal distance

We define as *ideal distance* of the ideal points A', B' (fig.5, theorem 9) the distance AB of the archetypes points of A', B' on the Euclidean line-archetype through the inversion.

But as from theorem 13 we have

AB=A'B'/OA'.OB' ....(1) so

«distance A′B′»= A′B′/OA′.OB′. …..(2)

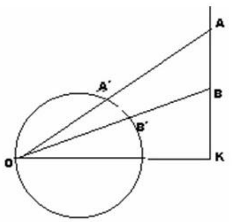


fig. 5

We must note here that the length A′B′/OA′.OB′. is not the Euclidean length of A' B' , but we agree to call it length, it is the ***ideal length***, the "length".

So the equal lengths in Euclidean plane remain equal in ideal plane!

The first condition of the distance is that "distance"AB="distance"AG+"distance"GB when G is between A and B that is easy to show.
The idea of "between-ness" is established from the fact that O is excepted from the ideal points of the ideal plane. If in figure 4 we had not except the point O then the points Γ_1 and Γ_2 should be between A and B. So we insure the axioms of order that are related with the second axiom of Euclid.

The 2^o and the 4^o axioms of Euclid in Ideal plane .
The definition of the *ideal distance* has in figure 6 this paradox: if we take on the ideal line the segment KA as unity of distances, then this unity becomes gradually smaller and smaller as we proceed along the line towards pole O, but the "distances" will remain "equals". So in figure 6 we should require infinite number of unities to reach at O, so we have the **second axiom of Euclid:** the "ideal length of the ideal line" is infinite, and there is not a last point on it, it is an open line.
For the fourth axiom we say: as in inversion the angles are preserved (conformal), the axiom for the right angles in Euclidean plane will also hold for the "right angles" in the Ideal plane" so the fourth Euclid's axiom may appear in the geometry of the Ideal Points and Lines.

For the 5^o axiom.

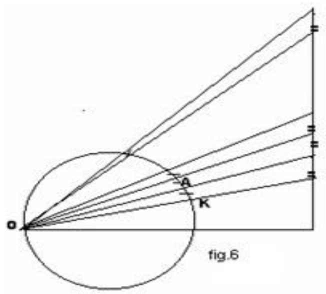


fig.6

Ideal parallel lines

If we have an Ideal line BΓ and an Ideal point A not on the line, we define the "*parallel Ideal line*" to BΓ, the circle which touches at O the circle coinciding with the given line, and also passes through the given point A. So the two Ideal lines touch each other at O , which is not an Ideal point, will be Ideal parallel lines and the second (**the circle is unique**) will passes through A. So the fifth axiom of Euclid holds on the Ideal straight lines.

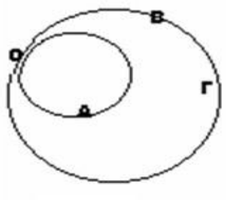


fig.7

For the 3^o axiom

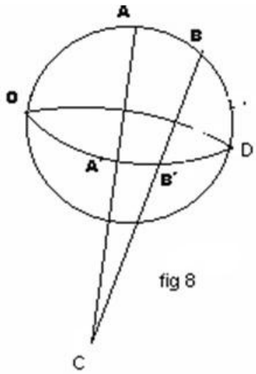


fig 8

Theorem 14 *the Ideal length of an Ideal segment is unaltered by inversion to any circle of the system.*
Let OD be any circle of the system and let C its centre. Then inversion changes an Ideal line into an Ideal line. Let the Ideal segment AB inverts into the Ideal segment A'B'. We can prove (formula 1) that AB and A'B' have the same Ideal length. (Bonola p.245)

Ideal displacements.
In the 3^o axiom in Euclid's plane, we have that the length of an linear segment must be unaltered as the segment will be displaced from one place of the plane to the other. So firstly we must make clean the concept of **plane displacement.**
We know that every plane displacement is a translation or a revolution or both. (γεωμετρία Καννέλου) But every translation or revolution can be resolved in infinite ways in two axial reflections, so the plane displacement will be studied from the concept of axial reflections. What is the Ideal reflection in an Ideal line?
Theorem 15 : *the inversion about any circle of the system in Ideal plane is equivalent to reflection of the Ideal points and lines in the Ideal line which coincides with the circle of inversion, and is orthogonal to the reflected Ideal line.*

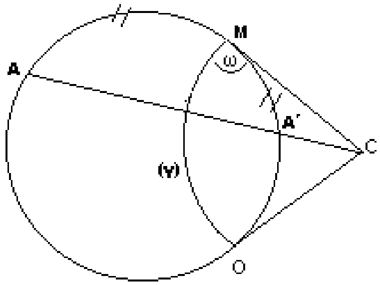


fig. 9

In figure 9, C is the centre of a circle (γ) of the system and A , A' inverse points with regard to this circle. Then the circle OAA' is orthogonal to the circle (γ) of inversion (theorem 8). This is the axis of the reflection we are going to prove. In other words A and A' are on the Ideal line perpendicular to the circle of inversion (ω=90). Also the Ideal line AA' is bisected by that circle in M , since the Ideal segment AM, inverts into the segment A'M , and Ideal lengths are unaltered by such inversion.(th.14) Such an inversion is, therefore the same as reflexion, and translation will occur when the circle of inversion (γ) is orthogonal to the given Ideal line.

The Ideal displacements are the results of Ideal Reflections.

So finally we have proved that any Ideal displacement of an Ideal line segment, does not alter it's Ideal length, so in Ideal geometry holds the third axiom of Euclid.

5. Ideal geometry is imaginary but mathematically consistent .

Finally we set up a one-to-one correspondence between the points of the Euclidean plane into itself, the inversion, having a new space of two dimensions, the Ideal plane. There the images of the Euclidean straight lines were a system of circles but they completed the axioms of Euclid. Then the two spaces are isomorphic. So it is possible to "translate" every proposition in the ordinary plane geometry into a corresponding proposition in this Ideal geometry. We have only to use the words Ideal points, Ideal lines, Ideal parallels etc. in the ordinary points, lines, parallels. But we can invent relations which were unknown in the systems of circles, as we see through the figure 10.

In figure 10 holds

In the (Ideal) triangle ABC the sum of the angles is π.

If ABC is rectangular and M the middle of BC then AM/OA.OM= ½ (BC/OB.OC)

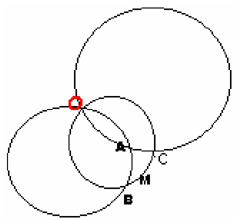


fig. 10

Yet $(AB^2/OA^2OB^2)+(AC^2/OA^2OC^2) = BC^2/OB^2OC^2$.
(Ideal Pythagorean theorem) etc.

6. Comment

The interpretation of the Ideal plane is not an interpretation of the world round us, as it is rejected by the observations, till now. The same happened with the Riemannian geometry of the curved space, whose curvature we never "saw", but we had to accept it as a reality if we want to interpret the reality of the conclusions of general relativity.

After the isomorphism through the inversion, the Euclidean straight line and geometry, loses it's absolute character supported from experience, and we have the beginning of the great revolution in mathematics, that of non-Euclidean geometries. The truth of the Ideal plane is an hypothetical truth, an Aristotelian form, in the realm of potential reality. The two isomorphic spaces changed the perspective of mathematics, separated them from the accepted set of initial statements (*material axiomatics*) which were linked with the intuition, and led to a deeper study and refinement of the axiomatic procedure (*formal axiomatics*).

Conclusions

How could the Ideal geometry be real? the straight line for us should be every circle in the figure 11. We must imagine we are tiny and every segment ds, looks as straight line. Also the rays of light, trace out the circles of the system, with the point O being a black hole. If we lived in such physical conditions our geometry should be Euclidean.

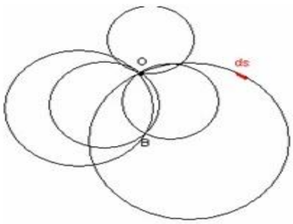


fig.11

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Convex geometric reasoning for crystalline energies *

Thaicia Stona
University of Sao Paulo
thaicia.stona@usp.br

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Abstract : The present work revisits the classical Wulff problem restricted to crystalline integrands, a class of surface energies that gives rise to finitely faceted crystals. The general proof of the Wulff theorem was given by J.E. Taylor (1978) by methods of Geometric Measure Theory. This work follows a simpler and direct way through Minkowski Theory by taking advantage of the convex properties of the considered Wulff shapes.

Introduction

This work is a short though sufficiently self-contained incursion into the Wulff construction and the Wulff theorem for faceted crystals, mathematically represented by the class of crystalline integrands.

The aim of the Wulff Problem is to find a surface whose total surface energy is minimal for a given fixed volume. This classical problem is also known as the Equilibrium Shape Problem, and the solution is also called an equilibrium shape, or simply a crystal. The problem is named after George Wulff, who invented an algorithm to determine the final shape of a crystal that grows near equilibrium, based on Josiah Willard Gibbs principle of the surface Gibbs free energy minimization for the evolution of a crystal droplet.

By convexifying γ, one induces a γ-metric on the dual space of the solution space. Since such a class of problems have polyhedral solutions, we can dismiss the geometric measure versions of Brunn-Minkowski Theorem

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from Federer and Wulff Theorem by applying the Legendre transform to the canonical version of the Wulff construction and build our way to the Convex Geometry version of Brunn-Minkowski Theorem through geometric inequalities and convexity. We show equivalences between constructions and some relations between the crystalline integrand and the area integrand version of the problem - isoperimetry and minimal surfaces.

I. The Crystalline Variational Problem

The Wulff shape arises in surface energy minimization problems when the energy function Φ is anisotropic. For isotropic energies and a given amount of mass, the equilibrium shape is well knows: a ball, formally denoted by the n-dimensional sphere S^{n-1}. This shape encloses the prescribed mass whereas minimizing the surface area of it. This is stated by the classic Isoperimetric Inequality.
For anisotropic energies, the analogous minimizer is the Wulff shape. Such energies have been heuristically misrepresented by simple functions that very often are not well-defined, presenting many singularities and unbounded energy spots. We avoid this imposture following Taylor's Geometric Measure Theory characterization of the energy. In this context, the surface energy, also called the energy function of the anisotropic problem, is an integrand, as defined below:

Def.:[integrand] An integrand on $\mathbb{R}^{n+1}$ is a function that will represent the surface energy function

$$\Phi : \mathbb{R}^{n+1} \times G_0(n+1, n) \to [0 , +\infty)$$

where the Grassmannian $G_0(n+1, n)$ is the manifold that parametrizes every n-dimensional linear subspace of $\mathbb{R}^{n+1}$, i.e., all the hyperplanes of the $n+1$-dimensional Euclidean space.

An integrand is defined constant coefficient iff

$$\Phi(x,\pi) = \Phi(p,\pi) \quad \forall x,\pi \in \mathbb{R}^{n+1} \ , \ \forall p \in G_0(n+1, n).$$

In this case, Φ is a function of its second variable only. An integrand is unoriented if it independs on the orientation of π . We will assume all integrands are continuous, constant coefficient and positively oriented.

Def.:[Wulff construction] Given an integrand Φ, plot it radially by taking each direction $v \in S^n$ and calculating Φ on the positively oriented plane π whose normal vector is $v, \pi = \left\{ x \in \mathbb{R}^{n+1} / \langle x, v \rangle = 0 \right\}$.We will denote π by

$v^{\perp}$ and vice versa. Plot $\Phi\left(v^{\perp}\right)$ in v direction: $\Phi\left(v^{\perp}\right)v$.

Then, for each v ,define the half-space $H_v \doteq \left\{x \in \mathbb{R}^{n+1} / \left\langle x,v\right\rangle \leq \Phi\left(v^{\perp}\right)\right\}$.

Take the intersection of all half-spaces. The resulting set W_Φ is the Wulff shape of Φ , also called the crystal of Φ :

$$W_\Phi \doteq \bigcap_{v \in S^n} H_v$$

For an isotropic energy, Φ =constant : the Wulff problem reduces to the Isoperimetric Inequality and the crystal is an Euclidean ball; that is the case of a soap bubble.

Obs.: We can extend homogeneously the function Φ in order to calculate it on other planes related to non unitary direction vectors by formalizing the explained abuse of notation defining the dual function Φ^* as follows

$$\Phi^* : S^n \to \left[0,+\infty\right) \quad , \quad \Phi^*\left(v\right) = \Phi\left(\pi\right)$$

where $v \perp \pi$ as defined above, $\Phi^*\left(p\right) \doteq \left|p\right| \Phi^*\left(\frac{p}{\left|p\right|}\right)$.

Note that since W_Φ is given by an intersection of half-spaces, then W_Φ is convex. Also we can assume $0 \in W_\Phi$ always. The physical meaning of the origin is the crystal seed for growing a crystal, a tiny monocrystal that induces the orientation of the new crystal.

Def.:[Legendre Transform] Let $\xi : S^{n-1} \to \mathbb{R}^+$ be a continuous function. The (first) Legendre transform of ξ is

$$\xi^*\left(v\right) \doteq \inf_{\left\langle \theta,v\right\rangle > 0} \frac{\xi\left(\theta\right)}{\left\langle \theta,v\right\rangle} \quad \text{where } \left|\theta\right| = 1$$

An alternative construction of the Wulff shape is based on the Legendre transform as in [5]:

Def.:[Fu's Wulff construction] Let W be the operator over integrands

$$W(\Phi)(\pi) \doteq \inf_{v\in S^n} \frac{\Phi^*(v)}{\left\langle \pi^{\perp}, v\right\rangle}$$

where $\left\langle \pi^{\perp}, v\right\rangle > 0$. Then the crystal of Φ is the set enclosed by the radial plot of $W(\Phi)$, plotted as explained before. Also, the orientation of W_Φ is defined positive.

Proposition: The two given definitions of crystal are equivalent.

Proof: Call Z the operator defined in Fu's construction instead of W :
$(W_\Phi \subset Z_\Phi)$: Let $y \in W_\Phi$, i.e. $\langle y, v\rangle \le \Phi^*(v)$ $(\forall v \in S^n)$. Then

$$\langle y, v\rangle = \frac{|y|}{|y|}\langle y, v\rangle = |y|\left\langle \frac{y}{|y|}, v\right\rangle \le \Phi^*(v)$$

If $|y|\left\langle \frac{y}{|y|}, v\right\rangle > 0$ then we have $|y| \le \dfrac{\Phi^*(v)}{\left\langle \frac{y}{|y|}, v\right\rangle}$. Since the inequality holds for arbitrary v, then

$$|y| \le \inf_{v\in S^n} \frac{\Phi^*(v)}{\langle y, v\rangle}, \quad \text{i.e. } y \in Z_\Phi$$

If $\left\langle \frac{y}{|y|}, v\right\rangle < 0$, then obviously the inequality holds, with y in the same half-space bounded by $\langle x, y\rangle = \Phi^*\left(\frac{y}{|y|}\right)$; $(Z_\Phi \subset W_\Phi)$:

Let $y \in Z_\Phi$, i.e. $|y| \le (Z(\Phi))^*\left(\frac{y}{|y|}\right)$, $\left\langle \frac{y}{|y|}, v\right\rangle > 0$.

Then :

$$|y| \le \frac{\Phi^*(v)}{\left\langle \frac{y}{|y|}, v \right\rangle} \quad \forall v \in S^n \quad \Leftrightarrow \quad y\left\langle \frac{y}{|y|}, v \right\rangle = \langle y, v \rangle \le \Phi^*(v)$$

for all $v \in S^n$ but then $y \in H_v \ \ (\forall v \in S^n) \Rightarrow \mathrm{y} \in W_\Phi$ □

II. Pathway through convexity

It is easy to visualize what kind of Wulff shape one gets when the intersection of half-spaces is finite: a polyhedron, except for unbounded and/or empty intersections. That is the case of anisotropic energies: we say that an integrand Φ is crystalline if its Wulff shape, or crystal W_Φ is a polyhedron. Now we take advantage of this fact:

Def.:[extreme point] Given a set $K \subset \mathbb{R}^n$, $x \in K$ is extreme if it cannot be expressed as a convex combination of any two other points of K .

Def.:[polytope] A polytope $P \subset \mathbb{R}^n$ is the convex hull of a finite set: $P = \left[\left\{ p_1, \mathrm{p}_2, \ldots, p_k \right\}\right]$.

Def.:[polar body] Given a convex set K , the polar body of K is the set $K^* \doteq \left\{ x \in \mathbb{R}^n \,/\, \langle x, y \rangle \le 1 \ \ (\forall y \in K) \right\}$.

Lemma: A supporting hyperplane H to a bounded convex set K contains at least one extreme point of K .

Proof: Denote the set of extreme points of K by E_K . Since K is convex, $K = [K]$, so $E_K \subset K \Rightarrow [E_K] \subset K$. We also have that $H \cap K = H \cap \partial K$, so the set of extreme points of $H \cap K$, $E_{H \cap K}$, is the set $E_{H \cap E_K}$. Now suppose the claim is true for every set with dimension $\leq m-1$. Then it is also true for all sets of dimension m, since if a given non-extreme point in m dimension could be written as a convex combination in dimension $m-1$, then it would be sufficient to write it in m dimension putting $\lambda_m = 0$. But for dimension 1, the claim is trivially true. Therefore it is true for any dimension. □

Theorem 1: A bounded convex set K is the convex hull of its extreme points.

Proof: Since $E_K \subset K \Rightarrow [E_K] \subset K$, we only need to prove that $K \subset [E_K]$.

Suppose some $x \in K$ is not in $[E_K]$. Then there exists a separating hyperplane H that separates strictly x from E_K. The parallel supporting hyperplane of K that is strictly separated from E_K by H must contain a point of E_K (lemma). Contradiction. □

Corollary: Every polytope is a finite intersection of half-spaces.

Proof: If P is finite, then so is E_P. For each $p \in E_P$, let A_P be the set of supporting hyperplanes that contains p and also contains at least another extreme point of P. Then take $A_p^{'} \subset A_p$ the subset that contains supp. hyperplanes intersecting the maximum number of extreme points as possible (this number is well-defined since the very # P is a majorant). The facets of P will be contained on those hyperplanes; for each facet define the half-space oriented to contain the origin and take the intersection of it. Because of the theorem, P is contained in this intersection. □

Theorem 2: If K is convex, then $K^{**} = K$.

Proof: $\left(K \subset K^{**}\right)$ Let $x \in K$. Then for any $y \in K^{*}$ we have $\langle x, y \rangle \leq 1$.

But then, since x is arbitrary, it has to be in K^{**}.

$\left(K^{**} \subset K\right)$ Let $y \in K^{**}$ and suppose $y \notin K$. Then there is a separating hyperplane H that separates y from K, $H = \left\{ x / \left\langle x,v \right\rangle = 1 \right\}$, $\left\langle x,v \right\rangle \le 1$ when $x \in K$ and $\left\langle y,v \right\rangle > 1$

But if $\left\langle x,v \right\rangle \le 1$ when $x \in K$, then $v \in K^*$ and $\left\langle y,v \right\rangle \le 1$ since $y \in K^{**}$.
Contrad iction . □

Theorem 2 reveals a link between Convex Geometry and Functional Analysis: given a polyhedral crystal W, we apply the corollary to define a convex Φ_C whose crystal coincides with W, so that Φ_C is the "smallest" enclosing function for W. For that, we use the theorem 2 by taking the polar of W . Since Φ is a linear operator, we know its behavior everywhere by homogeneous extension. By Riesz representation theorem, the crystal W is the polar of the unit ball $\Phi_C \equiv 1$ in the dual norm. That gives us the surface energy scaled so that the Wulff shape is given in units of surface free energy.

Def.:[Steiner symmetrization] For a convex body $K \subset \mathbb{R}^n$ and a $\theta \in S^{n-1}$, the Steiner symmetrization of K in the direction of θ is given by

$$S_\theta(K) \doteq \left\{ x + \lambda.\theta \middle| \ x \in \mathrm{Pr}oj_{\theta^\perp} \ K, \ \lambda \in \mathbb{R} \right\}$$

where $|\lambda| \le \frac{1}{2}\left|K \cap \left\{ x + \mathbb{R}\theta \right\}\right|$. Some properties are the fact that $\left|S_\theta(K)\right| = |K|$, $S_\theta(K)$ is convex and the convex Minkowski sum of symmetrizations equals to the symmetrization of the convex sum of the bodies. The symmetrization process slices K along θ , aligning the slices by putting their midpoints in $\theta^\perp$.

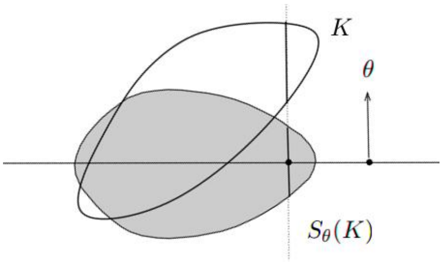

Figure 1: Example of Steiner symmetrization of K along the vector θ†

A useful classical result is stated below without its proof, which follows directly from the several interesting properties of the Steiner Symmetrization process. A more curious reader might refer to [6], [20] or [21].

Theorem: [Steiner-Schwarz] Given a convex body $K \subset \mathbb{R}^n$ and F a k-dimensional subspace, then there exists a sequence of symmetrizations θ_j such that the limiting body $\bar{K}$ satisfies $\left|\bar{K} \cap \{x+F\}\right| = \left|K \cap \{x+F\}\right|$, where $\bar{K} \cap \{x+F\}$ is a $k-$dimensional ball centered in x with radius $r(x)$.

Theorem: [**Brunn's Concavity Principle**] Given $K \subset \mathbb{R}^n$ a convex body and F a $k-$dimensional subspace of $\mathbb{R}^n$, the function $f : F^{\perp} \to \mathbb{R}^{+}$ given by

$f(x) = \left|K \cap \{x+F\}\right|^{\frac{1}{n}}$ is concave on its support.

Proof: Apply the former theorem and use that $\operatorname{supt} r(x) = \operatorname{Pr} oj_{F^{\perp}} K$,

$$f(x) = \left|\bar{K} \cap \{x+F\}\right| = Vol\left(S^{k-1}\right) = \frac{\pi^{2}}{\Gamma\left(\frac{k}{2}+1\right)} r(x)^{k}. \quad \square$$

† Figure adapted from [24]

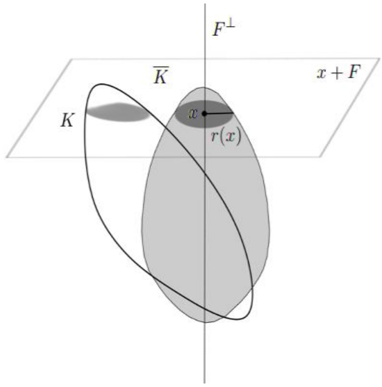

Figure 2: Application of Steiner-Schwarz to prove Brunn's Concavity Principle, where n = 3, k = 2 ††

The Brunn-Minkowski inequality is the crucial ingredient for proving the optimality of the Wulff shape. We conclude this section with a proof based on convex sum of two convex bodies and the Concavity Principle:

Theorem: [Brunn-Minkowski Inequality] Given non-empty compact subsets A, B of $\mathbb{R}^n$ $\quad |A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$

Proof: Take the Steiner symmetrization of A and B to find two convex bodies in $\mathbb{R}^n$. Create their convex sum L on $\mathbb{R}^{n+1}$ by taking the convex hull of $S_\theta(A)\times 0$ and $S_\theta(B)\times 1$, where $0, 1$ belong to the additional real axis for the convex sum, so that $L(t)=\left\{ x\in\mathbb{R}^n \,\middle|\, (x,t)\in L\right\}$.

Then $L\left(\frac{1}{2}\right)=\frac{S_\theta(A)}{2}+\frac{S_\theta(B)}{2}=\frac{S_\theta(A)+S_\theta(B)}{2}$. By the concavity principle applied for $F=\mathbb{R}^n$

$$\left|\frac{S_\theta(A)+S_\theta(B)}{2}\right|^{\frac{1}{n}} \geq \frac{1}{2}|S_\theta(A)|^{\frac{1}{n}}+\frac{1}{2}|S_\theta(B)|^{\frac{1}{n}} \qquad \square$$

†† Figure adapted from [24]

III. The Wulff Theorem

Wulff's 1901 seminal article provided a method to predict crystal shapes after Gibbs' proposition on the minimization of surface energy; since then, many have worked on the subject. Nevertheless, it was Taylor ([1]) who proved that the Wulff construction determines the unique minimizer W_Φ for the integral of Φ over the boundary ∂W_Φ. The proof requires some concepts from Geometric Measure Theory, which are now introduced:

Def.: [**integral current**] An integral n-dimensional current $S \subset \mathbb{R}^{n+1}$ is a rectifiable oriented hypersurface generalized through GMT so that eventual anomalous portions are still well-behaved enough to allow integration with respect to the measure $|S|$ on $\mathbb{R}^{n+1}$, which is a function of the Hausdorff measure H^n restricted to the support of S, which can be arbitrarily closely approximate by a n-d C^1 manifold. An interesting property of currents is that their boundaries also have the essential properties to allow boundary integration (for more see [3]). In the next theorem P will denote the current whose boundary is an integral current. The total surface energy of an integral current $S \subset \mathbb{R}^{n+1}$ is given by :

$$\Phi(S) \doteq \int_{x\in S} \Phi\left[n_S(x)\right] dH^n x$$

We also define for $h > 0$ the homothety in $\mathbb{R}^{n+1}$ $\mu_h(x) = hx$ and the integrand Φ the isomorphism $W_\Phi^h = \mu_{h\#}(W_\Phi)$ following [1].

Theorem:[**Wulff**] Given an integrand Φ, then for every n-dimensional current $P \subset \mathbb{R}^{n+1}$

$$\Phi(\partial W_\Phi) \le \Phi(\partial P).$$

up to translations and homotheties, such that their mass coincide, $M(P) = M(W_\Phi)$

Proof: Let P be a current with ∂P its positively oriented, piecewise C^1 boundary. Then

$$\Phi(\partial P)=\int \Phi\left(\overrightarrow{\partial P}(x)\right) d\,|\partial P|\,x \geq \int \operatorname{supt}\left(W_{\Phi}\right)\left(\overrightarrow{\partial P}(x)\right) d\,|\partial P|\,x$$

$$=\lim_{h \to 0} \frac{M\left(P^{h}\right)-M(P)}{h}$$

where $M\left(W_{\Phi}\right)=M(P)$ and $M\left(W_{\Phi}^{h}\right)=h^{n+1} M\left(W_{\Phi}\right)$ and P^{h} is the positively oriented current given by the Minkowski sum $x+y$ where $x \in \operatorname{supt} P$ and $y \in \operatorname{supt} W_{\Phi}^{h}$. Then Brunn-Minkowski inequality implies:

$$\lim_{h \to 0} \frac{M\left(P^{h}\right)-M(P)}{h} \geq \lim_{h \to 0} \frac{(1+h)^{n+1} M\left(W_{\Phi}\right)-M\left(W_{\Phi}\right)}{h}$$

$$=\lim \frac{\left(1+h^{n+1}-1\right)}{h} M\left(W_{\Phi}\right)=\lim_{h \to 0} \frac{M\left(W_{\Phi}\right)}{h} \sum_{i=1}^{n+1} h^{i} . \frac{(n+1)!}{i!\,(n+1-i)!}$$

$$=\lim_{h \to 0} M\left(W_{\Phi}\right).(n+1)=(n+1).M\left(W_{\Phi}\right)$$

In particular for $P \in W_{\Phi}$, the above inequalities are equalities. By using the fact that $M(P) \in M\left(W_{\Phi}\right)$, we conclude that $\Phi\left(\partial W_{\Phi}\right) \leq \Phi(\partial P)$. Such shape is unique modulo translations and homotheties, and since the mass is fixed, follows the uniqueness of W_{Φ}. □

IV. Conclusion

In this exposition, different fundamental areas of Mathematics were gathered to structure a simple mathematical basis for the equilibrium shape problem with a crystalline integrand. A natural generalization of the Wulff construction for non-equilibrium growth is to replace the energy function for the correspondent potential that controls the process, the mobility function. Also, through Kinetic PDEs, a flourishing area of mathematical modeling in the Sciences, it might be of interest to study the growth and the stability of such shapes.

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پست الکترونیک : CJCMEmail@gmail.com , smaraissian53@gmail.com
تماس : (+98)911 143 4988 , (+98) 13 - 42613646

ویراستاری و صفحه بندی و تنظیم : موسسه فرهنگی و هنری ره پویان اندیشه و هنر
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