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In the name of God

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Contents

Exact and numerical soliton solutions to nonlinear wave equations

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Abstract: Because most physical systems are inherently nonlinear by nature, mathematical modeling of physical systems often leads to nonlinear evolution equations. Investigating traveling wave solutions of nonlinear evolution equations (NLEEs) plays a significant role in studying such nonlinear physical phenomena. This paper discusses the application of the functional variable method for finding solitary and periodic wave solutions of the longitudinal wave motion NLEE in a nonlinear magneto-electro-elastic (MEE) circular rod. Each of the obtained solutions

contains an explicit function of the variables in the considered equations. The applied method yielded a powerful mathematical tool for solving these nonlinear wave equations without the necessity of a computer algebra system; according the widespread view "real mathematicians don not compute". However, if the obtained exact hyperbolic solutions are executed numerically, then they show an undesirable unknown numerical conditioning problem. This paper contains a warning against it and information about how to prevent this outcome.

INTRODUCTION

Understanding of a wide variety of nonlinear phenomena in different branches of physics and other knowledge domains has been achieved by means of exact integrable model equations, which solving yields a deeper understanding than those models approximately describing nature. Of course, "nearly" integrable systems and their evolution can be described by means of perturbation theory; building upon wave dynamics. But this paper focuses on constructing exact soliton waves and the periodic solutions for nonlinear evolution equations, including their subsequent exploitation for numerical purposes. This study was originated by Korteweg, together with PhD candidate, de Vries. Together, they derived, from physics first principles, a nonlinear partial differential equation, known as the KdV equation.

Solutions are best understood by studying waves in shallow waters. The KdV equation revealed solitary waves (i.e. waves with permanent shape). This seemingly contradicts nature, as a traveling wave commonly dissolves by radiation, as in the electromagnetic case. The permanency of shape is due to both a balance of dispersion and nonlinear effects. The nonlinear effects derive from wave packets, of nearly the same length propagating with their group velocity, while individual component waves in the packet move through the packet with their phase velocity. It can be generally shown that the energy of a wave disturbance is propagated at the group velocity, not at the phase velocity.

Long-wave components of a general solution travel faster than the shortwave components, thus the components disperse. The standard linear theory predicts the dispersal of any disturbance other than a purely sinusoidal one.

Zabusky and Kruskal (both in 1965 at Bell Labs in Murray Hill, New Jersey, USA) simulated numerical solutions of the KdV equation [1]. They noticed, using a difference analog for the KdV equation [2; p. 233], that colliding solitary waves of emerged from collisions with all of their original properties intact. They also noted sinusoidal waves evolving into solitary waves with dispersive radiation. Because of this particle-like behavior of the solitary waves, they coined the term "*solitons*" [3, 4]. The name caught on quickly amongst mathematical physicists as the soliton was the first "particle" to be discovered with pure mathematics.

Incidentally, in water tank experiments, conducted in 1834, John Scott Russell observed that solitary waves could pass through each other and subsequently emerge unchanged. As the concept of fundamental particles did not emerge until early in the next century, Russel did conceptualize his findings as such. Nowadays, the phenomenon is viewed from a more general point of view; recognizing solitons and their approximations in many wave phenomena.

The superconducting state is not the sole exception; there are also topological solitons known as vortices and fluxions. On both microscopic and macroscopic levels solitons are discoverable. Taking into account the omniscient inherent nonlinearities in the nature of matter and propagation media, appearance of solitons can be predicted. This paper provides a solution for the nonlinear equation of motion for magnetic excitation and confirms the existence of magnetic solitons. Of course, under strict mathematical definition, which implies an infinite life-time and an infinity of conservation laws, exact solitons cannot exist. The mathematical definition of solitons, albeit derived from nature by Zabusky and Kruskal, is never fully observable as in nature there is not an infinite lifetime. The observed 'quasi-solitons' are often so long-lived that they are almost 'true solitons'. This is why physicists often generalize the word 'soliton', without full justice to mathematical rigor [5]. Physics, Chemistry, and Biology describe processes and phenomena using theoretical Differential Models endowed with empirical parameters and/or empirical functions. Exact solutions of those differential equations are preferable for researchers as they enable the design of experiments yielding either a positive or negative correlation via creating approximating natural conditions, from which estimates are derived as to their parameters and/or functions. This makes the search for exact solutions of NLEEs very significant.

The insolvability of model equations, however grounded they empirically are, is potentially a severe drawback. This is where the invention of mathematical methods is critical as it provides exact solutions to the model equations. As a result, many techniques of solving traveling wave equations have been developed over the last three decades, such as, the Tanh–Coth function method [6, 7]; the Kudryashov method [8]; the Exp-function method [9 - 12]; the homotopy perturbation method [13 - 15]; the modified simple equation method [16 - 19]; the (G'/G)-expansion method [20 - 24]; the exp-expansion method [25]; the transformed rational function method [26]; Ricatti Ansätze [27]; the multiple exp-function method [28, 29]; the generalized Hirota bilinear method [30]; and the Frobenius integrable decompositions [31]; as well as many others. Each methods has its distinct pluses and minuses.

A unified method applicable to all types of NLEEs would be a scientific break-through for all knowledge domains, including Physics. Until then, improvement of a particular method to reveal/predict unknown solutions to existing NLEEs, will have to be researched in the various application domains.

Zerarka et al. [32] in 2010 proposed a functional variable method to solve NLEEs arising in mathematical physics. This pioneering work spawned studies to refine the initial idea [5, 33]
This paper applies the functional variable method to construct exact solutions for nonlinear longitudinal wave motion equations in nonlinear magneto-electro-elastic circular rods. We extended the use of the functional variable method by developing a novel solution procedure of simplicity, capable of extension to all NLEEs.

The paper is organized as follows: Section 2, the functional variable method; Section 3, application of this method to the nonlinear evolution equation mentioned before; Section 4, results and discussion; and Section 5 Conclusions.

THE ALGORITHM OF THE FUNCTIONAL VARIABLE METHOD

Zerarka et al. [32] were the first to propose the functional variable method to solve various NLEEs arising in mathematical physics and engineering. First, the NLEE is written in two independent variables *x* and *t*,

$$F\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots\right) = 0 \tag{1}$$

as where the subscript denotes partial derivative, where F is a polynomial in $u(x,t)$ and its partial derivatives and $u(x,t)$ is called a dependent variable or unknown function, to be determined. The main steps of this method are as follows [5, 32-34]:

Step 1. To find the traveling wave solutions of Eq.(1) we introduce the wave variable $\xi = k(x \pm \omega t)$, so that $u(x,t) = u(\xi)$, where $\omega \in \Re - \{0\}$ is the wave velocity and k the wave number. This, reduces Eq.(1) into the following ordinary differential equation (ODE):

$$X\left(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \cdots\right) = 0 \tag{2}$$

Where X is a polynomial in $u(\xi)$ and its derivatives, $u_\xi = \frac{du}{d\xi}, u_{\xi\xi} = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. We make a transformation in which the unknown function $u(\xi)$ is considered as a functional variable in the form:

$$u_\xi = Y(u) \tag{3}$$

and some successively derivatives of $u(\xi)$ are as follows:

$$u_{\xi\xi} = Y\,Y' = \frac{1}{2}\left(Y^2\right)'$$
$$u_{\xi\xi\xi} = \frac{1}{2}\left(Y^2\right)'' Y = \frac{1}{2}\left(Y^2\right)'' \sqrt{Y^2}$$
$$u_{\xi\xi\xi\xi} = \frac{1}{2}\left((Y^2)''' Y^2 + \frac{1}{2}\left(Y^2\right)''\left(Y^2\right)'\right) \tag{4}$$

… … … … …

where $Y' = \frac{dY}{du}, Y'' = \frac{d^2Y}{du^2}$ and so on.

Step 3. We substitute Eq.(3) and Eq.(4) into Eq.(2) to reduce it to the following ODE:

$$Q\left(u, Y, Y', Y'', \cdots\cdots\cdots\right) = 0 \tag{5}$$

Step 4: The key idea of Eq.(5) is of special importance because it admits analytical solutions for a large class of nonlinear wave equations. After integration, Eq.(5) provides an expression of Y, and this, in turn with (3), gives appropriate solutions to the original wave equations. In order to illustrate how the method works, we examine some examples in the following section, were previously been treated using other pre-existing methods.

APPLICATION OF THE FUNCTIONAL VARIABLE METHOD

Nowadays in solid mechanics, nonlinear elastic effects on solitary waves are receiving considerable attention. On the basis of classical linear theory, Zhang and Liu [35] solved the nonlinear equations for a thin elastic rod and derived the longitudinal, torsional and flexural waves using Hamilton's variation principle. Liu and Zhang [36] solved the nonlinear wave equation in an elastic rod by the Jacobi elliptic function expansion method.
With the increasing usage of magneto-electro-elastic (MEE) structures in various engineering fields, such as actuators, sensors, etc., wave propagation in MEE media has also attracted more researchers. Using the propagator matrix and state-vector approach, Chen et al. [37] presented an analytical treatment for the propagation of harmonic waves in MEE multilayered plates.
Based on the constitutive relation for transversely isotropic piezoelectric and piezomagnetic materials, combined with the differential equations of motion, Xue et al. [38] derived a longitudinal wave motion equation in a Magneto-electro-elastic circular rod of the form,

$$u_{tt} - c_0^2 \cdot u_{zz} - \left(\frac{c_0^2}{2} u^2 + N u_{tt} \right)_{zz} = 0 \tag{6}$$

where c_0 is the linear longitudinal wave velocity for a MEE circular rod and N is the dispersion parameter, both depending on the material properties as well as the geometry of the rod. Eq.(6) is a nonlinear wave equation with dispersion caused by the transverse Poisson’s effect.

The traveling wave transformation is:

$$u(x,t) = u(\xi), \qquad \xi = k\left(z - \omega t\right) \tag{7}$$

Eq.(7) reduces Eq.(6) into the following ODEs:

$$k^2\omega^2 u_{\xi\xi} - k^2 c_0^2 \cdot u_{\xi\xi} - k^2\left(\frac{c_0^2}{2}u^2 + Nk^2\omega^2 u_{\xi\xi}\right)_{\xi\xi} = 0 \tag{8}$$

By integrating Eq.(8) twice with respect to ξ and neglecting the integration constant, we obtain:

$$u_{\xi\xi} = \frac{\omega^2 - c_0^2}{Nk^2\omega^2}u - \frac{c_0^2}{2Nk^2\omega^2}u^2 . \tag{9}$$

Following Eq.(4), it is easy to deduce from Eq.(9) an expression for the function $Y(u)$:

$$\left(Y^2(u)\right)' = \frac{2\left(\omega^2 - c_0^2\right)}{Nk^2\omega^2}u - \frac{c_0^2}{Nk^2\omega^2}u^2 . \tag{10}$$

Integrating Eq.(10) and setting the constant of integration to zero yields:

$$Y^2(u) = \frac{\left(\omega^2 - c_0^2\right)}{Nk^2\omega^2}u^2 - \frac{c_0^2}{3Nk^2\omega^2}u^3 \tag{11}$$

or

$$Y(u) = \pm\frac{c_0}{\omega\, k}\sqrt{\left(\frac{1}{3\mathrm{N}}\right)}\, u\sqrt{A-u}\ , \text{ where } A = \frac{3\left(\omega^2 - c_0^2\right)}{c_0^2}. \tag{12}$$

Substituting Eq.(3) into Eq.(12), we obtain

$$u_\xi = \pm \frac{c_0}{\omega k}\sqrt{\left(\frac{1}{3\mathrm{N}}\right)}\, u\sqrt{A-u}\ .$$

(13)

Separating the variables in Eq.(13) and then integrating, by setting the constant of integration to zero, yields:

$$\int \frac{du}{u\sqrt{(A-u)}} = \pm \frac{c_0}{\omega k}\sqrt{\left(\frac{1}{3\mathrm{N}}\right)}\,\xi\,.$$

(14)

Finally, by completing the integration in Eq. (14), two cases of traveling wave solutions of the longitudinal motion equation are obtained (in a nonlinear magneto-electro-elastic circular rod) after a straightforward algebraic manipulation.

If $\frac{\left(\omega^2 - c_{0^2}\right)}{N} > 0$ we obtain the following hyperbolic traveling wave solutions $u(z,t)$:

$$u_1(z,t) = \frac{3\left(\omega^2 - c_0^2\right)}{c_0^2}\operatorname{sech}^2\left(\frac{1}{2\omega}\sqrt{\frac{\left(\omega^2 - c_0^2\right)}{N}}\left(z - \omega\, t\right)\right).$$

(15)

$$u_2(z,t) = -\frac{3\left(\omega^2 - c_0^2\right)}{c_0^2}\operatorname{csch}^2\left(\frac{1}{2\omega}\sqrt{\frac{\left(\omega^2 - c_0^2\right)}{N}}\left(z - \omega\, t\right)\right)$$

(16)

Eq.(15) and Eq.(16) are solitary wave solutions for the longitudinal wave motion equation of a magneto-electro-elastic circular rod.

If $\frac{\left(\omega^2 - c_0^2\right)}{N} < 0$, Eq. (15) and Eq. (16) reduce to periodic wave solutions as follows:

$$u_3(z,t) = \frac{3\left(\omega^2 - c_0^2\right)}{c_0^2}\sec^2\left(\frac{1}{2\omega}\sqrt{\left(\frac{c_0^2 - \omega^2}{N}\right)}\left(z - \omega\, t\right)\right)$$

(17)

$$u_3(z,t) = \frac{3\left(\omega^2 - c_0^2\right)}{c_0^2}\csc^2\left(\frac{1}{2\omega}\sqrt{\left(\frac{c_0^2 - \omega^2}{N}\right)}\left(z - \omega\, t\right)\right)$$

(18)

Eq.(17) and Eq.(18) are periodic wave solutions for the longitudinal wave motion equation of a magneto-electro-elastic circular rod. ***Note***: In Eq. (15) - Eq. (18), $\omega \neq \pm c_0$.

Remark**:** The validity and reliability of the obtained results was checked with help of Maple, by substituting them back into the original equations and found them to be correct. Numerical accuracy, however, in the results Eq. (15) - Eq. (18) may impede in applications. This depends on the default accuracy of the number representation of the numerical computation engine. For example, in Eq. (16), the large *x* the right hand side (RHS) exhibited much better numerical accuracy than the left hand side (LHS). Depending on the computer's decimal digits representation d, inaccuracy may represent a major factor.

Consequently, care has to be taken in determining which representation to choose for industrial applications. If it is mandatory to have well-conditioned numerical evaluations and the arguments for the circular or hyperbolic functions are low or high, then choose either our un-simplified results Eq. (15) - Eq. (18), or the following symbolic simplifications of our results Eq. (19) - Eq. (22):

$$u_1(z,t) = \frac{2\mathrm{A}}{1 + \cosh\left(\frac{c_0}{\omega}\sqrt{\frac{A}{3\mathrm{N}}}\left(z - \omega\, t\right)\right)}$$

(19)

$$u_2(z,t) = \frac{2A}{1 - \cosh\left(\frac{c_0}{\omega} \sqrt{\frac{A}{3N}} \left(z - \omega\, t \right) \right)} \tag{20}$$

$$u_3(z,t) = \frac{2A}{1 + \cos\left(\frac{c_0}{\omega} \sqrt{\frac{-A}{3N}} \left(z - \omega\, t \right) \right)} \tag{21}$$

$$u_4(z,t) = \frac{2A}{1 - \cos\left(\frac{c_0}{\omega} \sqrt{\frac{-A}{3N}} \left(z - \omega\, t \right) \right)} \tag{22}$$

Note the similarity of the hyperbolic identities Eqns.(19, 20) with the circular Eqns. (21, 22).

The achieved mathematical or 'symbolic' correctness disguises a common numerical conditioning problem, omnipresent in the literature. It escapes awareness because solutions such as the those yielded by KdV equations are usually given as a squared hyperbolic function.

For small parameters just these squared hyperbolic functions are much worse than the reduced-order equivalents Eq. (19), Eq. (20) presented herein. In conclusion, reduction of the order of solutions obtained here remarkably improves the numerical accuracy, for small arguments.

RESULTS AND DISCUSSION

Now we will discuss the wave features of our obtained solutions. The obtained solutions Eq.(15) and Eq.(16) are solitary waves and Eq.(17) and Eq.(18) are plane periodic waves. The wave amplitude is $A=\frac{3\left(\omega^2-c_0^2\right)}{c_0^2}$, the wave number is $k=\frac{1}{2\omega}\sqrt{\frac{\left(\omega^2-c_0^2\right)}{N}}$ and the wave length is $\lambda=\frac{2\pi}{k}=\frac{4\,\omega\pi}{c_0}\sqrt{\frac{3N}{A}}$. For solitary waves to exist $\omega > c_0$ and the wave length is inversely proportional to the square root of the amplitude A, i.e., $\lambda \propto \frac{1}{\sqrt{A}}$. On the other hand, if $\omega < c_0$ the waves are periodic and also traveling.

Solitary waves can be obtained from each traveling wave solution by setting particular values to its unknown parameters. By adjusting these parameters, one can get an internally localized mode. We present some Maple-plots of solitary waves constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equation. In Figures1- 4.

The Eq. (15) exhibits bell-shaped soliton solutions. It has infinite wings or tails. This soliton is referred to as non-topological. This solution does not depend on the amplitude and high frequency. Figure 1 shows the shape of the exact bell-shaped soliton solution i.e., non-topological soliton solution of the MEE equation. (The figure only shows the shape of Eq. (15) with wave speed $\omega=2$, $c_0=1$ and $N=1$ within the constraints -3 $\leq z, t \leq 3$).

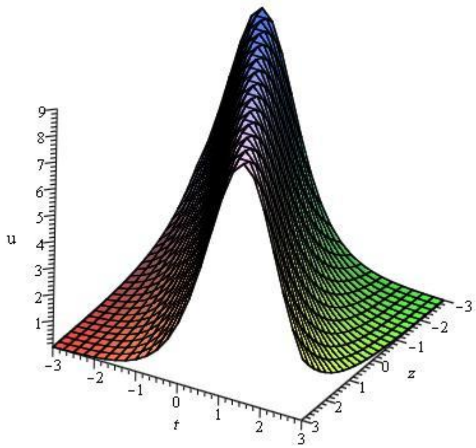

Fig. 1: Bell profile of Eq.(15) for wave speed *ω* = 2, c_0 = 1 and *N* = 1, within the constraints -3 ≤ *z, t* ≤ 3.

The Eq.(16) is the singular soliton solutions of the MEE equation. Figure 2 shows the shape of singular solitons of Eq.(16) for a wave speed of $\omega = 2$, $c_0 = 1$ and $N = -2$ within the constraints 3 ≤ *z*, *t* ≤ 3.

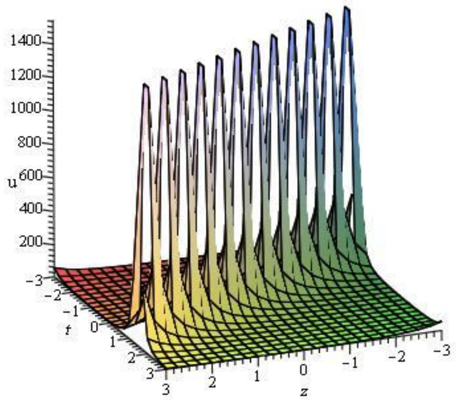

Fig. 2: Singular soliton of Eq.(16) for wave speed *ω* = 2, c_0 = 1 and *N* = -2, within the constraints -3 ≤ *z, t* ≤ 3.

Eq.(17) conveys the periodic traveling wave solutions of the MEE equation. Figure 3 shows the shape of these for wave speed $\omega = 2$, $c_0 = 1$ and $N = 1$ within the constraints $-3 \leq z, t \leq 3$.

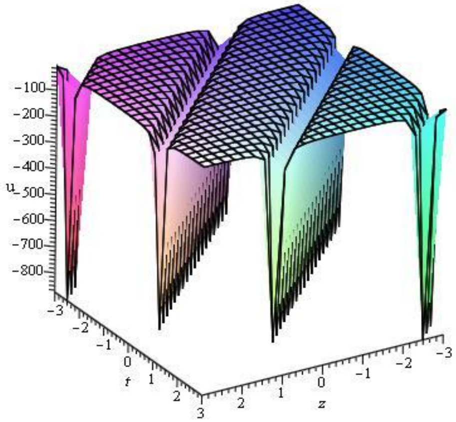

Fig. 3: Periodic profile of Eq.(17) for wave speed *ω* = 2, c_0 = 1 and *N* = 1, within the constraints -3 ≤ *z, t* ≤ 3.

Eq.(18) is the periodic traveling wave solutions of the MEE equation. Figure 4 shows the shape of periodic solution of Eq.(18) for a wave speed of $\omega=2$, $c_0 = 1$ and $N = -2$ within the constraints $-3 \leq z, t \leq 3$.

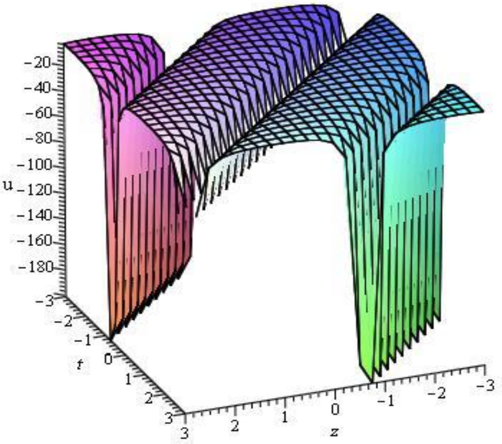

Fig. 4: Singular profile of Eq.(18) for wave speed $\omega = 2$, $c_0 = 1$ and $N = -2$, within the constraints $-3 \leq z, t \leq 3$.

CONCLUSIONS

By using the functional variable method we obtained solitary and periodic wave solutions of longitudinal wave motion equations of a nonlinear magneto-electro-elastic circular rods. The consistency of the method and the reduction of computational effort demonstrate its wider applicability. The solution procedure is very simple and the traveling wave solutions are expressed by hyperbolic, and trigonometric functions. The results also show that the method is simple and effective, and can be applied to many other nonlinear wave equations arising in mathematical physics.

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A Survey on Effective Solution of the Boundary Value Problem for Improperly Elliptic Equations

S. M. Ali Raeisian

Abstract
An effective method for solving Riemann type problem for Improperly elliptic equations in complex plane is presented. We reduce the problem to a number of boundary value problems for properly elliptic equations, which can be solve by grid method.

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Keyword : boundary value problem, improperly elliptic equation, grid method .

Introduction

Let D be a simply connected domain in a complex plane with boundary $\Gamma = \partial D$, and consider the equation **[5]**

$$\frac{\partial^n u}{\partial \bar{z}^q \, \partial z^p} \equiv \frac{1}{2^n} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right)^q \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right)^p = 0, \qquad (x, y) \in D, \; z = x + iy, \tag{1}$$

$(n = p + q)$, with boundary conditions

$$\left.\frac{\partial^k u}{\partial N^k}\right|_\Gamma = f_k(x,y), \quad k=0,\dots,p-1, \qquad (x,y)\in\Gamma, \qquad (2\,a)$$

$$\left.\Re\left(\frac{\partial^k u}{\partial N^k}\right)\right|_\Gamma = f_k(x,y), \; k=p,\dots,q-1, \qquad (x,y)\in\Gamma\,. \qquad (2\,b)$$

without loss of generality, we suppose $q>p$; that is (1) known as an Improperly Elliptic Equation **[5]** . Last $q-p$ boundary functions are real.

Here, we will reduce the Riemann problem for the improperly equation (1) $(2\,a),(2\,b)$, to a uniquely solvable Dirichlet problems for properly elliptic equations; to determine real and imaginary part of the solution separately.

By assuming $U=\Re(u)=\frac{u+\bar{u}}{2}$ and using $(2\,a),(2\,b)$ we get that the function U satisfies the problem

$$\frac{\partial^{2q}U}{\partial z^q \partial \bar{z}^q}=0 \;, \qquad z\in D$$

$$\left.\frac{\partial^k U(x,y)}{\partial N^k}\right|_\Gamma = \Re(f_k), \qquad (k=0,\dots,q-1), \; (x,y)\in\Gamma \qquad (3)$$

This problem is a Dirichlet problem for q -harmonic equation which is uniquely solvable, so it remain to find $\Im u$.

We represent the solution of the problem (1) - $(2\,a,b)$ as

$$u=V+i\;w, \; V=U+iW \qquad (4)$$

where w is the real valued function, satisfying the condition $\Delta^p w = 0$.

From $(2a)$ we have

$$\Im \frac{\partial^k u}{\partial N^k} = \Im f_k , \qquad (k = 0,..., p-1),$$

$$\frac{\partial^k (\Im u)}{\partial N^k} = \Im f_k, \qquad (k = 0,..., p-1)$$

On the other hand from (4) we have : $\Im u = w + \Im V = w + W$.

By substituting, we get

$$\Im \frac{\partial^k V}{\partial N^k} + \frac{\partial^k w}{\partial N^k} = \Im f_k$$

i.e.

$$\frac{\partial^k w}{\partial N^k} = \Im f_k - \Im\left(\frac{\partial^k V}{\partial N^k}\right).$$

Now, by solving Dirichlet problem for P - harmonic equation

$$\frac{\partial^{2p} w}{\partial z^p \partial \overline{z}^p} = 0 \ , \qquad z \in D$$

$$\left.\frac{\partial^k w(x, y)}{\partial N^k}\right|_{\Gamma} = \Im f_k (x, y) - \Im\left(\frac{\partial^k V (x, y)}{\partial N^k}\right) \ (x, y) \in \Gamma, \ \ (k = 0,..., p-1)$$

We will find the solution w. We see, that the functions U and w are determined uniquely, but the function W is determined uniquely up to $(q-p)^2$ real constants. So for the uniqueness of the solution we must add $(q-p)^2$ complementary conditions.

The method was applied for second , and third order equations in a rectangle **[1] , [4]** , and was experimentally tested .

Now , we consider the elliptic equation

$$\frac{\partial^5}{\partial\overline{z}^3\,\partial z^2}u(x,y)\equiv\frac{1}{2^5}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^3\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^2=0\quad(x,y)\in D\ . \tag{5}$$

We take following boundary conditions for problem (5)

$$u\big|_\Gamma=f_0\ ,\ \frac{\partial u}{\partial N}\bigg|_\Gamma=f_1\ ,\ \Re\left(\frac{\partial^2 u}{\partial N^2}\right)\bigg|_\Gamma=f_2\ , \tag{6}$$

where $f_0\in C^{(2,\alpha)}(\Gamma)$, $f_1\in C^{(1,\alpha)}(\Gamma)$, $f_2\in C^{(\alpha)}(\Gamma)$ (i.e. a function f_0 with second order derivative $\frac{d^2 f_0}{ds^2}$ satisfy Holder condition on Γ, f_1 with first order derivative satisfy Holder condition on Γ and f_2 satisfy Holder condition on Γ) are given functions on Γ. We are seeking the solution of $(5),(6)$ in the class of functions $C^{(5)}(D)\cap C^{(2,\alpha)}(D\cup\Gamma)$.

The general solution of (1) can be represented in the form:

$$u=\overline{z}^2\ \varphi_0(z)+\overline{z}\,\tilde{\varphi}_1(z)+z\ \tilde{\varphi}_2(\overline{z})+\tilde{\varphi}_3(z)+\tilde{\varphi}_4(\overline{z}),$$

where $\varphi_0,\tilde{\varphi}_i\ \ (1\le i\le 4)$ are arbitrary analytic functions in D.

We may replace

$$\tilde{\varphi}_1(z)=C_1+z\,\varphi_1(z)\ ,\ \tilde{\varphi}_2(\overline{z})=C_2+\overline{z}\,\varphi_2(\overline{z}),$$

So, we get into

$$u=\overline{z}^2\,\varphi_0(z)+C_1\overline{z}+z\,\overline{z}\,\varphi_1(z)+C_2z+z\,\overline{z}\,\varphi_2(\overline{z})+\tilde{\varphi}_3(z)+\tilde{\varphi}_4(\overline{z})\ \ ,$$

$$u=\overline{z}^2\varphi_0(z)+z\overline{z}\left(\varphi_1(z)+\varphi_2(\overline{z})\right)+\left(\Phi_3(z)+\Phi_4(\overline{z})\right)\ , \qquad (7)$$

We denote $U=\Re u$, we get

$$\frac{\partial^6 U}{\partial z^3\,\partial\overline{z}^3}=\frac{\partial^6\left(\frac{u+\overline{u}}{2}\right)}{\partial\overline{z}^3\,\partial z^3}=\frac{1}{2}\left(\frac{\partial^6 u}{\partial\overline{z}^3\,\partial z^3}+\frac{\partial^6\overline{u}}{\partial\overline{z}^3\,\partial z^3}\right)=0\ \ .$$

We have a Dirichlet problem for the determination of U :

$$\begin{aligned}
&\Delta^3 U=0,\\
&U\big|_\Gamma=\Re(f_0)\big|_\Gamma,\\
&\left.\frac{\partial U}{\partial N}\right|_\Gamma=\Re(f_1)\big|_\Gamma,\\
&\left.\frac{\partial^2 U}{\partial N^2}\right|_\Gamma=\Re(f_2)\big|_\Gamma\ \ ,
\end{aligned}\qquad (8)$$

The solvability and smoothness of the solution of problem (8) follows from the general theory of elliptic problems **[3] .** From the unique solution of Dirichlet problem for third harmonic equation (8) we have U on all mesh points and a formula for representing unique solution of above Dirichlet problem may be found in **[2 , p.149].**

Applying bi-harmonic operator Δ^2 on real part of u in (7); we get into

$$
\begin{aligned}
2\Delta^2 U &= 2\Delta\left[\Delta\left(\overline{z}^2\varphi_0(z)+z^2\overline{\varphi_0(z)}\right)\right] \\
&= \Delta\left(2\overline{z}\varphi_0'(z)+2z\overline{\varphi_0'(z)}+\left(z\Phi(z)\right)'+\overline{\left(z\Phi(z)\right)'}\right) \\
&= 2\varphi_0''(z)+2\overline{\varphi_0''(z)} = 4\Re\left(\varphi_0''(z)\right),
\end{aligned}
$$

Therefore

$$\Delta^2 U = 2\Re\left(\varphi_0''(z)\right) , \tag{9}$$

where φ_0'' is analytic in D, so $\Re\varphi_0''(z)$ is harmonic function.

From (9) we have

$$\frac{1}{2}\Delta^2 U = \Re\left(\varphi_0''(z)\right) = \frac{\partial}{\partial x}\left(\Re\varphi_0'(z)\right) = \frac{\partial}{\partial y}\left(\Im\varphi_0'(z)\right) , \tag{10}$$

Denoting $u_1 = \Re\left(\varphi_0'(z)\right)$ and $v_1 = \Im\left(\varphi_0'(z)\right)$, we have Poincare problems for the determination of these functions

$$\Delta u_1 = 0,$$
$$\left(\frac{\partial u_1}{\partial x}\right)\Big|_{\Gamma} = \frac{1}{2}\left(\Delta^2 U\right), \tag{11}$$

and

$$\Delta v_1 = 0,$$
$$\left(\frac{\partial v_1}{\partial y}\right)\Big|_{\Gamma} = \frac{1}{2}\left(\Delta^2 U\right), \tag{12}$$

Solving these problems, we get: $u_1 = u_1^0 + c_1 y + c_2$ and $v_1 = v_1^0 + c_3 x + c_4$, where u_1^0, v_1^0 are uniquely determined functions and c_j $\left(j = 1,..4\right)$ are arbitrary real constants.

We must mention that by Cauchy – Riemann equations we have $c_1 = -c_3$, therefore, we get a representation

$$\varphi_0' = \Phi^0 + C_1 - i\,C_0 z \ ,$$

Here C_1 is complex and C_0 is real arbitrary constant and $\Phi^0 = u_1^0 + i v_1^0$ is uniquely determined analytic function. By integration, we have

$$\varphi_0\left(z\right) = w_0\left(z\right) + i\ C_0\ z^2 + C_1\, z + C_2 \ , \tag{13}$$

where C_0 is real constant, C_1, C_2 are arbitrary complex constants, and w_0 is uniquely determined function .

Now, replacing φ_0 in $\left(7\right)$, we get

$$u = \left(\bar{z}^2 w_0 + i C_0 \left(z\bar{z}\right)^2 + C_1\, z\bar{z}^2 + C_2\, \bar{z}^2\right) + z\bar{z}\left(\varphi_1\left(z\right) + \varphi_2\left(\bar{z}\right)\right) + \left(\Phi_3\left(z\right) + \Phi_4\left(\bar{z}\right)\right)$$

$$u = \bar{z}^2 w_0 + iC_0 (z\bar{z})^2 + z\bar{z}\left(C_1\bar{z} + \varphi_1(z) + \varphi_2(\bar{z})\right) + \left(C_2\bar{z}^2 + \Phi_3(z) + \Phi_4(\bar{z})\right)$$

$$u = \bar{z}^2 w_0 + iC_0 (z\bar{z})^2 + \left[z\bar{z}\left(\Re\left(\bar{C}_1 z + \varphi_1(z) + \overline{\varphi_2(\bar{z})}\right) + i\,\Im\left(-\bar{C}_1 z + \varphi_1(z) - \overline{\varphi_2(\bar{z})}\right)\right)\right]$$

$$\left[\left(\Re\left(\overline{C_2}\, z^2\right) - i\,\Im\left(\overline{C_2}\, z^2\right)\right) + \left(\Re\Phi_3(z) + i\,\Im\Phi_3(z)\right) + \left(\Re\Phi_4(z) - i\,\Im\Phi_4(z)\right)\right]$$

$$u = \bar{z}^2 w_0 + iC_0 (z\bar{z})^2 + \left[z\bar{z}\left(\Re\left(\bar{C}_1 z + \varphi_1(z) + \overline{\varphi_2(\bar{z})}\right) + i\,\Im\left(-\bar{C}_1 z + \varphi_1(z) - \overline{\varphi_2(\bar{z})}\right)\right)\right]$$

$$+\left[\left(\Re\left(\overline{C_2}\, z^2 + \Phi_3(z) + \Phi_4(z)\right) + i\,\Im\left(-\overline{C_2} z^2 + \Phi_3(z) - \Phi_4(z)\right)\right)\right]$$

i.e.

$$u = \bar{z}^2 w_0 + iC_0 (z\bar{z})^2 + \left(z\bar{z}\ \Re\Omega_1(z) + \Re\Omega_2(z)\right) + i\left(z\bar{z}\ \Im\Omega_3(z) + \Im\Omega_4(z)\right), \tag{14}$$

where

$$\Omega_1(z) = \bar{C}_1 z + \varphi_1(z) + \overline{\varphi_2(\bar{z})},\ \Omega_2(z) = \overline{C_2}\, z^2 + \Phi_3(z) + \Phi_4(z),$$

$$\Omega_3(z) = -\bar{C}_1 z + \varphi_1(z) - \overline{\varphi_2(\bar{z})},\ \Omega_4(z) = \left(-\overline{C_2} z^2 + \Phi_3(z) - \Phi_4(z)\right)$$

are arbitrary analytic functions.

Finally, we represent the solution in the form

$$u=\overline{z}^2\, w_0\left(z\right)+i\,C_0\left(z\overline{z}\right)^2+H\left(z,\overline{z}\right)+i\,h\left(z,\overline{z}\right)\ , \tag{15}$$

where $H\left(z,\overline{z}\right)=z\overline{z}\ \Re\Omega_1\left(z\right)+\Re\Omega_2\left(z\right)$ and $h\left(z,\overline{z}\right)=z\overline{z}\ \Im\Omega_3\left(z\right)+\Im\Omega_4\left(z\right)$. Here w_0 is known function, and H,h are real valued functions which satisfy the condition $\Delta^2 H=\Delta^2 h=0$ and constant C_0 is arbitrary real constant.

Now, we must determine real valued functions H,h. These functions satisfy bi-harmonic equation: $\Delta^2 H=\Delta^2 h=0$ and from (15)

$$U=\Re\left(\overline{z}^2\, w_0\left(z\right)\right)+H\left(x,y\right),$$

Hence, we have following Dirichlet conditions on the boundary Γ

$$H\big|_\Gamma=\Re\left(f_0\right)-\Re\left(\overline{z}^2\, w_0\left(z\right)\right)\Big|_\Gamma \quad ,$$

$$\frac{\partial H}{\partial N}\bigg|_\Gamma=\Re\left(f_1\right)-\frac{\partial}{\partial N}\Re\left(\overline{z}^2\, w_0\left(z\right)\right)\bigg|_\Gamma\ ,$$

Finally we get the Dirichlet problem for bi-harmonic equation

$$\Delta^2 H=0\ ,$$

$$H\big|_\Gamma=\Re\left(f_0\right)-\Re\left(\overline{z}^2\, w_0\left(z\right)\right)\Big|_\Gamma\ , \tag{16}$$

$$\frac{\partial H}{\partial N}\bigg|_\Gamma=\Re\left(f_1\right)-\frac{\partial}{\partial N}\Re\left(\overline{z}^2\, w_0\left(z\right)\right)\bigg|_\Gamma\ ,$$

which has a unique solution.

Analogously, we get

$$\Im u = \Im\left(\overline{z}^2\, w_0(z)\right) + C_0\,(z\overline{z})^2 + h(x,y)\,,$$

i.e.

$$h(x,y) = \Im u - \Im\left(\overline{z}^2\, w_0(z)\right) + C_0\,(z\overline{z})^2\,, \qquad (17)$$

and we have the same boundary problem for determination of function h. This problem includes arbitrary constant C_0, therefore must be modified .

So, first we find the function h_0, by solving Dirichlet problem for bi-harmonic equation

$$\begin{aligned} &\Delta^2 h_0 = 0\,, \\ &h_0\big|_\Gamma = (z\overline{z})^2\Big|_\Gamma\,, \\ &\left(\frac{\partial}{\partial N} h_0\right)\bigg|_\Gamma = \left(\frac{\partial}{\partial N}(z\overline{z})^2\right)\bigg|_\Gamma\,, \end{aligned} \qquad (18)$$

and then, we solve following problem

$$\begin{aligned} &\Delta^2 h_1 = 0, \\ &h_1\big|_\Gamma = (\Im u)\big|_\Gamma - \Im\left(\overline{z}^2\, w_0(z)\right)\Big|_\Gamma = \Im f_0 - \Im\left(\overline{z}^2 w_0(z)\right)\Big|_\Gamma\,, \\ &\left(\frac{\partial h_1}{\partial N}\right)\bigg|_\Gamma = \left(\frac{\partial}{\partial N}\Im u\right)\bigg|_\Gamma - \frac{\partial}{\partial N}\Im\left(\overline{z}^2 w_0(z)\right)\Big|_\Gamma = \Im f_1 - \left(\frac{\partial}{\partial N}\Im\left(\overline{z}^2 w_0(z)\right)\right)\bigg|_\Gamma\,, \end{aligned} \qquad (19)$$

These problems are uniquely solvable ,and the solution of (19) will be $h = h_1 - C_0\, h_0$.

Replacing the function h into (15), we find

$$u=\overline{z}^2 w_0(z)+H(x,y)+i\left(h_1(x,y)-C_0 h_0(x,y) \right)+ i C_0 (z\overline{z})^2 ,$$

or

$$u=\overline{z}^2 w_0(z)+H(x,y)+i h_1(x,y)+ i C_0\left((z\overline{z})^2-h_0 \right),$$

So, during above argument we had proved that by using our algorithm we can find the solution of the problem (5), (6) with only one constant C_0, and linearly independent solution for corresponding homogeneous problem is

$$u_0=i\left((z\overline{z})^2-h_0 \right),$$

Here h_0 is bi-harmonic function and uniquely determined from problem (18).

Thus, applying the same method as in previous points, we reduce the start problem to the boundary value problems for properly elliptic equations with real coefficients.

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