بسم الله الرحمن الرحیم

# In the name of God

**Caspian Journal of Computational & Mathematical Engineering (CJCME)**

**Director & Licensee : Dr. Seyed Mohammadali Raeisian**
**Editor in Chief : Dr. S.M.Ali Raeisian**
**License No : 75729**
**ISSN (Print) : 2476 – 4418 ,**
**ISSN (Online) : 2476 – 5252**

**Email address : CJCMEmail@gmail.com**
**Postal address : P.O .Box 44815 -1165 ( Dr.S.M.Ali Raeisian ), Roodsar, Gilan , Iran**
**Phone : (+98) 13-42 613646 , (+98) 911 143 4988**
**Website : https://caspianjcme.com**

**Editing & Paging : Rah-poyan Andisheh & Honar cultural & artistic institute.**
**rpandishehonar@gmail.com**

**Price ( Inside of Iran ) : 200 000 IRR**
**Price ( Outside of Iran + Shipping cost ) : 20 $**

**March 2017 , No.1**

# Submission Guidelines

## Manuscript

- Paper must be current and their main contribution have not been published elsewhere .

- The title of the paper must concise and informative . It is highly desirable to divide the paper into different sections , such as Introduction , Problem Statements , Results , Discussion , and Conclusion .

- Presented papers must include an abstract of maximum 70 words .

. Your article(s) must be classified with MSC 2010 .

. The manuscript must be prepared and submitted as a Word file *(Maximum 20 pages )* .

. The name(s) , affiliation(s) , address(es) , and email(s) must be included .

- You must submit the paper electronically to journal email address *CJCMEmail@gmail.com*

## Review process

- Article should be submitted to the related *CJCME* reviews team and undergo rigorous peer review .

- Your name , address , and email will be seen by reviewers .

• The journal has permission to accept , reject , remove , or summarize some part of the manuscript for publication .

## Copy right

Knowing the fact that the journal publishes results of academic and research activities of faculty members and researchers in all engineering and fundamental sciences fields free of any publication fee , papers are considered on the understanding that , if they are accepted for publication , the entire copy right shall pass to CJCME journal . All authors are asked to sign a copy right agreement to this effect . The completed form with your final revised paper must be submitted . Any paper published in this journal may be republished in print or in another online venue .

# Contents



**Caspian Journal of Computational & Mathematical Engineering**
**2017 , No.1 (March 2017)**
**https://caspianjcme.com**

# Numerical Solution of Fractional Neutral Functional-Differential Equations by the Operational Tau Method

**S. Kouhkani,**
Department of Mathematics, Islamic Azad University branch of Shabestar, Shabestar, Iran
skouhkani@yahoo.com

**H. Koppelaar**
Fac.of Electrical Eng., Mathematics and Computer Science, University of Delft, Netherlands
Koppelaar.henk@gmail.com

**M. Abri**
Department of Computer, Islamic Azad University branch of Shabestar, Shabestar, Iran
mhsn.abri@gmail.com

**AMS 2010 Mathematical Subject Classification**: Primary: 26A33; Secondary: 34K28, 34K37, 34K40, 41A50.

***Keywords*:** operational Tau method, fractional equations, fractional neutral functional differential equations.

**DOI :** http://dx.doi.org/10.22039/cjcme.2017.01

**Abstract:** We develop an extension of the algebraic formulation of the Operational Tau Method (OTM) based upon shifted Chebyshev polynomials. This extension enables us to improve numerical precision of solving Fractional Neutral Functional-Differential equations (FNFDEs) from Bhrawy and Algamdi [8] as we show in this paper.

## 1. INTRODUCTION

The tau method was introduced by Lanczos [1] in 1938 for solving ordinary differential equations. In 1981, Ortiz and Samara [2] presented a new approach to the tau method by proposing an operational technique for the numerical solution of a single nonlinear ordinary differential equation with some supplementary conditions.

The main advantage of the operational tau method based on shifted Chebyshev polynomials is to reduce the fractional neutral functional-differential equation to a set of algebraic equations. The main advantage of the OTM is its simplicity, and is more convenient for computer algorithms.

During the last thirty years considerable work has been done in the development of this technique, its theoretical analysis and numerical applications [3], for numerical solution of many problems such as partial differential equations, Integral equations, Integral-differential equations and so on [4].

Mathematical folklore sets the birth of the concept of fractional calculus in the year 1695 by the answer to a question raised by L'Hôpital(1661-1704) to Leibniz (1646-1716), in which he sought the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ for derivativesif $n = \frac{1}{2}, \frac{1}{3}, \ldots$. In his reply, dated 30 September 1695, Leibniz wrote to L'Hôpital(quoting from [5]) "this is an apparent paradox from which, one day, useful consequence will be drawn…".

The first book devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [5]published in 1974. A much later book, by Podlubny [6], is from 1996 and the book by Kilbas, Srivastava and Trujillo [7] appeared in 2006.

In recent years, it has turned out that many phenomena in viscoelasticity, fluid mechanics, biology, chemistry, acoustics, control theory, psychology and other areas of science can be successfully modeled by the use of fractional order derivatives [9].

The objective of this paper is to develop an extension of the OTM such that equations are solved with improved precision. This is exemplified by taking problems from Bhrawy and Alghamdi[8].

## 2. DEFINITIONS

In this section, we state definitions of fractional calculus [8-10], as needed in the sequel.

**Definition 1.**

The Riemann-Liouville fractional integral operator of order θ (θ > 0) is defined as

$$J^{\theta} f(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-s)^{\theta-1} f(s) ds, \qquad \theta > 0, t > 0 \qquad (2.1)$$

$$.J^0 f(t) = f(t)$$

Here Γ is the Gamma function. Some of the most important properties of operator $J^{\theta}$ for $f(t)$ , are as follows

i) $J^{\theta} J^{\alpha} f(t) = J^{\theta+\alpha} f(t)$

ii) $J^{\theta} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\theta+\gamma+1)} t^{\theta+\gamma}$(2.2)

iii) $J^{\theta} J^{\alpha} f(t) = J^{\alpha} J^{\theta} f(t)$

(2.2)

**Definition 2.**

The Caputo fractional derivatives of order $\theta$ of $f(t)$ are defined as

$$D^{\theta} f(t) = J^{m-\theta} D^m f(t) = \frac{1}{\Gamma(m-\theta)} \int_0^t (t-s)^{m-\theta-1} \frac{d^m}{ds^m} f(s) ds \ , \qquad (2.3)$$

$$t > 0 \, , \qquad m-1 < \theta < m$$

where $D^m$ is the classical differential operator of order m . Also

$$D^{\theta} C = 0 \qquad (C \text{ is a constant}),$$

$$D^{\theta} t^{\mu} = \begin{cases} 0 & \mu \in N_0 \, , \quad \mu < \lceil \theta \rceil \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\theta)} t^{\mu-\theta}, & \mu \in N_0 \, , \mu \geq \lceil \theta \rceil \ or \ \mu \notin N \, , \quad \mu > \lfloor \theta \rfloor \end{cases}$$

(2.4)

## 3. THE OPERATIONAL TAU METHOD

Here, we state some relevant properties of the tau method. We define the scalar product $\langle\, ,\rangle$, for any integrable functions ψ(t) and Φ(t) on [a,b], by $\langle\psi(t),\Phi(t)\rangle_\omega = \int_a^b \psi(t)\Phi(t)\omega(t)dt$,where $\|\psi\|_\omega^2 = \langle\psi(t),\psi(t)\rangle_\omega$ and $\omega(t)$ is a weight function. Let $L_\omega^2[a,b]$ be the space of all functions $f:[a,b]\to\mathbb{R}$, with $\|f\|_\omega^2 < \infty$.

The main idea of the method [10] is to approximate $u(t)\in L_\omega^2[a,b]$. Let $\Phi_t = \{\Phi_i(t)\}_{i=0}^{\infty} = \Phi T_t$ be a set of arbitrary orthogonal polynomial bases defined by a lower triangular matrix Φ and $T_t = [1,t,t^2,\dots]^*$.

The tau method is designed to convert linear or non-linear differential equations, delay differential equations or a system of these equations to a system of linear algebraic equations based on these three simple matrices

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \\ & \vdots & & & \ddots \end{bmatrix}, \eta = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \\ 0 & 0 & 3 & 0 & \\ & \vdots & & & \ddots \end{bmatrix}, p = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & \\ 0 & 0 & 0 & 0 & \\ & \vdots & & & \ddots \end{bmatrix}.$$

In the sequel we need operators on polynomials, stated in a lemma as follows.

**Lemma 1**.

Suppose that $u(t)$ is a polynomial as $(t) = \sum_{i=0}^{\infty} u_i t^i = uT_t$ , then we have [4]

$$D^r u(t) = \frac{d^r}{dt^r}u(t) = u\eta^r T_t \qquad r = 0,1,2,3,\dots$$

(3.1)

$$t^s u(t) = u\mu^s T_t \qquad s = 0,1,2,3,\dots$$

(3.2)

$$\int_a^x u(t)dt = upT_x - upT_a \qquad p \text{ is matrix } p.$$

(3.3)

For a proof see [4]. ■
Let us consider

$$u(t) = \sum_{i=0}^{\infty} u_i \Phi_i(t) = \mathrm{u}\Phi T_t \quad (3.4)$$

to be an orthogonal series expansion of the solution, where $\mathrm{u} = \{u_i\}_{i=0}^{\infty}$ is a vector of unknown coefficients, $\Phi T_t$ is an orthogonal basis for polynomials in $R$ [4].

## 4. SHIFTED ORTHOGONAL POLYNOMIALS

Chebyshev and Legendre polynomials are interesting examples of polynomials bases defined by a matrix. In this section we define shifted Chebyshev polynomials.

The Chebyshev polynomials are defined on $[-1,1]$ with

$$\begin{cases} T_0(x) = 1\,, T_1(x) = x, \\ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x)\,, \quad i = 1,2,3,\dots \end{cases}$$

$$[T_0(x), T_1(x), \mathrm{T}_2(x), \dots]^* = TX_x$$

$$.\ where\ T = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ -1 & 0 & 2 & 0 & \dots \\ 0 & -3 & 0 & 4 & \\ & \vdots & & & \end{bmatrix}, X_x = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \end{bmatrix}$$

and shifted Chebyshev polynomials $T^{\circ}$ on $x \in [a,b], a < b$, are defined as

$$\begin{cases} T_0^{\circ}(x) = 1\,, T_1^{\circ}(x) = \dfrac{2x-(b+a)}{b-a} & x\epsilon[a,b] \\ T_{i+1}^{\circ}(x) = 2\left(\dfrac{2x-(b+a)}{b-a}\right)T_i^{\circ}(x) - T_{i-1}^{\circ}(x), & i = 1,2,3,\dots \end{cases}$$

## 5. THE OPERATIONAL TAU METHOD FOR FNFDE AT WORK

Consider the following Fractional Neutral Functional Differential Equation from [8]

$$D^{\theta}\big(u(t) + a(t)u(p_m t)\big)$$

$$= \beta u(t) + \textstyle\sum_{n=0}^{m-1} b_n(t) D^{\gamma_n} u(p_n t) + f(t), \qquad t \ge 0 \tag{5.1}$$

with the initial conditions$\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i$.

Here $a,\ b_n(n = 0, 1, 2, \dots, m-1)$ are given analytical functions,

$-1 < \theta \le m$ , $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \theta$ and $\beta,\ p_n,\ c_{in},\ \lambda_i$ denote given constants with $0 < p_n < 1\ (n = 0, 1, \dots, m)$.

Now we apply the tau method and use the following process. By equations (2.3), (3.1) and (3.4), we obtain

$$D^{\theta}u(t) = J^{m-\theta}D^m(u\Phi T_t) = J^{m-\theta}(u\Phi\eta^m T_t) = u\Phi\eta^m J^{m-\theta}(T_t). \tag{5.2}$$

By definition1 and relation (iii) we get

$$J^{m-\theta}(T_t) = [J^{m-\theta}(1), J^{m-\theta}(t), \dots, J^{m-\theta}(t^{\gamma}), \dots]^* = [\tfrac{\Gamma(1)t^{m-\theta}}{\Gamma(m-\theta+1)}, \tfrac{\Gamma(2)t^{m-\theta+1}}{\Gamma(m-\theta+2)}, \dots, \tfrac{\Gamma(\gamma+1)t^{m-\theta+\gamma}}{\Gamma(m-\theta+\gamma+1)}, \dots]^* = \acute{\Gamma}\Pi \tag{5.3}$$

where $\acute{\Gamma}$ is an infinite diagonal matrix with elements

$$,\ \Pi = [t^{m-\theta},\ t^{m-\theta+1},\ \dots,\ t^{m-\theta+\gamma}, \dots]^*. \acute{\Gamma}_{ii} = \frac{\Gamma(i+1)}{\Gamma(m-\theta+i+1)}, \quad i = 0,1,2,\dots$$

By decomposing $t^{m-\theta+\gamma}$ $(\gamma = 0, 1, 2, \dots)$ with Chebyshev polynomials

$$t^{m-\theta+\gamma} = \sum_{i=0}^{\infty} a_{\gamma+i}\, \Phi(t) = a_\gamma\, \Phi T_t \;, a_\gamma = [a_{\gamma,0},\; a_{\gamma,1},\; \dots]$$

(5.4)

we obtain with

$\Pi = [a_0 \Phi T_t \,, a_1 \Phi T_t \,, \dots, a_\gamma \Phi T_t \,, \dots\,]^* = A\Phi T_t$ ,

(5.5)$A = [a_0 \,, a_1 \,, \dots, a_\gamma \,, \dots\,]^*$

Then, by substituting equation (5.3) in equation (5.2), we have

$$D^\theta u(t) = u\Phi\, \eta^m\, \acute{\Gamma}\Pi = u\Phi\eta^m\, \acute{\Gamma} A\Phi T_t = uV\Phi T_t\,, \quad V = \Phi\eta^m \acute{\Gamma} A$$

(5.6)

For delay functions, we have

$$u(p_m t) = u\Phi T_{p_m}(t), \text{ with} T_{p_i}(t) = [1, p_1(t), p_2(t), \dots .]^*.$$

(5.7)

By approximating each $p_m(t)$as $p_m(t) = \sum_{j=0}^{\infty} p_{mj} t^j$, we obtain

$$T_{p_m(t)} = \begin{bmatrix} 1 \\ p_{10} + p_{12}t + \cdots + p_{1n}t^n + \cdots \\ p_{20} + p_{21}t + \cdots + p_{2n}t^n + \cdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \end{bmatrix}$$

If we let $\Delta_i$ to be the last coefficient matrix, then$T_{p_m}(t) = \Delta_i T_t$. By substituting this in equation (5.7) we have

$$u(p_m t) = u\Phi\Delta_i T_{p_m}(t).$$

(5.8)

By approximating $a(t)$ and $b_n(t)$ with polynomials $a(t) = \sum_{k=0}^{n} a_{ik} t^k$, $b_n(t) = \sum_{k=0}^{n} b_{ik} t^k$,

we get, with help of $\tilde{a} = \sum_{k=0}^{n} a_{ik} u\Phi\Delta_i \mu^k \eta^m \acute{\Gamma} A\Phi$, the result

$D^{\theta}\big(a(t)u(p_m t)\big) = \sum_{k=0}^{n} a_{ik} t^k u\Phi\Delta_i \eta^m J^{m-\theta} T_t = \sum_{k=0}^{n} a_{ik} u\Phi\Delta_i \mu^k \eta^m J^{m-\theta} T_t =$
$\sum_{k=0}^{n} a_{ik} u\Phi\Delta_i \mu^k \eta^m \acute{\Gamma} A\Phi T_t = \tilde{a} T_t \,,$ (5.9)

And also, with $\tilde{b} = \sum_{n=0}^{m-1}\sum_{k=0}^{n} b_{ik} u\Phi\Delta_n \mu^k \eta^m \acute{\Gamma} A\Phi$, we obtain

$$\sum_{n=0}^{m-1} b_n(t) D^{\gamma_n} u(p_n t) = \sum_{n=0}^{m-1}\sum_{k=0}^{n} b_{ik} t^k u\Phi\Delta_n \eta^m J^{m-\gamma_n} T_t$$
$$= \sum_{n=0}^{m-1}\sum_{k=0}^{n} b_{ik} u\Phi\Delta_n \mu^k \eta^m J^{m-\gamma_n} T_t =$$

$\sum_{n=0}^{m-1}\sum_{k=0}^{n} b_{ik} u\Phi\Delta_n \mu^k \eta^m \acute{\Gamma} A\Phi T_t = \tilde{b} T_t \,,$ (5.10)

For $f(t)$ we have $f(t) = fT_t$ . Also for $\beta u(t)$ where $\beta$ is a constant and $\tilde{\beta} = \beta u\Phi$ we have

$\beta u(t) = \beta u\Phi T_t = \tilde{\beta} T_t,$ (5.11)

where $\tilde{\beta}$ is a diagonal matrix with elements $\beta$.

Now we substitute equations (5.8)-(5.11) into equation (5.1)

$$uV\Phi T_t + \tilde{a}\Phi^{-1}\Phi T_t = \tilde{\beta}\Phi^{-1}\Phi T_t + \tilde{b}\Phi^{-1}\Phi T_t + f\Phi^{-1}\Phi T_t$$

We rewrite the residual matrix $R(x)$ with $\tilde{R} = \big[uV + \tilde{a}\Phi^{-1} - \tilde{\beta}\Phi^{-1} - \tilde{b}\Phi^{-1} - f\Phi^{-1}\big]$ into

$$R(t) = [uV\Phi T_t + \tilde{a}\Phi^{-1}\Phi T_t - \tilde{\beta}\Phi^{-1}\Phi T_t - \tilde{b}\Phi^{-1}\Phi T_t - f\Phi^{-1}\Phi T_t]$$
$$= uV + \tilde{a}\Phi^{-1} - \tilde{\beta}\Phi^{-1} - \tilde{b}\Phi^{-1} - f\Phi^{-1}]\Phi T_t = \tilde{R}\Phi T_t$$

Setting the residual matrix equal to zero by $\tilde{R} = 0$, or applying orthogonality of the inner products $\langle R(x), \emptyset_k(x)\rangle_\omega = 0,\ k = 0, 1, \ldots$ then by the supplementary conditions in equation (5.1) we have

$\sum_{n=0}^{m-1} c_{in} u\eta^n \Phi T_0 = \lambda_i$ .

Thus we obtained an infinite algebraic system that is easily solvable.

## 6. COMPUTATION OF THE ERROR FUNCTIONS

In this section, an error estimation for the approximate solution of equation(5.1) with supplementary conditions is obtained. Let us call $e_n(t) = u(t) - u_n(t)$ the error function of the Tau approximants $u_n(t)$ to $u(t)$ where $u(t)$is the exact solution of equation (5.1). Therefore, $u_n(t)$ satisfies the following equations

$$D^{\theta}\big(u_n(t) + a(t)u_n(p_m t)\big) = \beta u_n(t) + \sum_{n=0}^{m-1} b_n(t) D^{\gamma_n} u_n(p_n t) + f_n(t) + H_n(t), \quad (6.1)$$

$$\sum_{n=0}^{m-1} c_{in} {u_n}^{(n)}(0) = \lambda_i,\ t \geq 0\,, -1 < \theta \leq m\,,\ 0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \theta.$$

The perturbation term $H_n(t)$ can be obtained by substituting the computed solution $u_n(t)$ into the equation

$$H_n(t) = D^{\theta}\big(u_n(t) + a(t)u_n(p_m t)\big) - \beta u_n(t) - \sum_{n=0}^{m-1} b_n(t) D^{\gamma_n} u(p_n t) - f(t).$$

We proceed to find an approximation $e_{n,N}(t)$ to the error function $e_n(t)$ in the same way as we did before for the solution of equation (5.1). Note that N is the degree of approximation of $u_n(t)$. By subtracting equation (6.1) from equation (5.1), we have

$$\begin{aligned} D^{\theta}((u(t) - u_n(t)) + a(t)(u(p_m t) - u_n(p_m t))) &= \beta(u(t) - u_n(t)) \\ &+ \sum_{n=0}^{m-1} b_n(t) D^{\gamma_n}\big(u(p_n t) - u_n(p_n t)\big) + \big(f(t) - f_n(t)\big) - H_n(t), \end{aligned}$$

$$\sum_{n=0}^{m-1} c_{in}(u^{(n)}(0) - {u_n}^{(n)}(0)) = 0,$$

Or

$$D^{\theta}\big(e_n(t) + a(t)(e_n(p_m t)\big) = \beta(e_n(t)) + \sum_{n=0}^{m-1} b_n(t) D^{\gamma_n}\big(e_n(p_n t)\big) - H_n(t),$$

$\sum_{n=0}^{m-1} c_{in}({e_n}^{(n)}(0)) = 0.$

It should be noted that in order to construct the approximate $e_{n,N}(t)$to $e_n(t)$, only the related equations (3.4) through (6.1) needs to be recomputed and the structure of the method remains the same [3-10].

## 7. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section some numerical examples are given to clarify the high accuracy of the method.

**Example 1.**

Consider the following FNFDE from [8]

$$u^{\frac{5}{2}}(t) = u(t) + u^{\frac{1}{2}}\left(\frac{t}{2}\right) + u^{\frac{3}{2}}\left(\frac{t}{3}\right) + \frac{1}{2}u^{\frac{5}{2}}\left(\frac{t}{4}\right) + \frac{\Gamma(5)}{\Gamma(\frac{5}{2})}\left(t^{\frac{3}{2}}\right) - \frac{\Gamma(4)}{\Gamma\left(\frac{3}{2}\right)}t^{\frac{1}{2}} - t^4$$
$$+ t^3 - \frac{\Gamma(5)}{\Gamma\left(\frac{9}{2}\right)}(\frac{t}{2})^{\frac{7}{2}} + \frac{\Gamma(4)}{\Gamma\left(\frac{7}{2}\right)}(\frac{t}{2})^{\frac{5}{2}} - \frac{\Gamma(5)}{\Gamma\left(\frac{7}{2}\right)}(\frac{t}{3})^{\frac{5}{2}} + \frac{\Gamma(4)}{\Gamma\left(\frac{5}{2}\right)}(\frac{t}{3})^{\frac{3}{2}}$$
$$- \frac{\Gamma(5)}{2\Gamma\left(\frac{5}{2}\right)}(\frac{t}{4})^{\frac{3}{2}} + \frac{\Gamma(4)}{2\Gamma\left(\frac{3}{2}\right)}(\frac{t}{4})^{\frac{1}{2}}, \quad t \in [0,1]$$

With

$$u(0) = 0, \qquad u^{'}(0) = 0, \quad u^{''}(0) = 0,$$

The exact solution is $u(t) = t^4 - t^3$.

The result for the approximate solution by OTM and the Exact solution for this example are given in Table1.

**Example 2.**

Consider the following FNFDE from [8]

$$u^{\frac{1}{2}}(t) = -u(t) + \frac{1}{4}u\left(\frac{t}{3}\right) + \frac{1}{3}u^{\frac{1}{2}}\left(\frac{t}{3}\right) + g(t), \quad u(0) = 1, \quad t \in [0,5]$$

Where

$$g(t) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}e^s ds + e^t - \frac{1}{4}e^{\frac{t}{3}} - \frac{1}{3\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}e^{\frac{s}{3}} ds,$$

and the exact solution is $u(t) = e^t$.

The result for the approximate solution by the OTM and the exact solution for example 2 are given in Table2.

In Table 3, we list the absolute errors obtained by the OTM.

**Table 1. OTM and the Exact Solution for Example 1**

| t | $u_{OTM}$ | $u_{Exact}$ |
|---|---|---|
| **0.1** | -0.000900000000013 | -0.000900000000000 |
| **0.2** | -0.006399999997823 | -0.006400000000000 |
| **0.3** | -0.018899999882211 | -0.018900000000000 |
| **0.4** | -0.038399999963938 | -0.038400000000000 |
| **0.5** | -0.062499999999480 | -0.062500000000000 |
| **0.6** | -0.086399999928974 | -0.086400000000000 |
| **0.7** | -0.102899999992440 | -0.102900000000000 |
| **0.8** | -0.102399999993932 | -0.102400000000000 |
| **0.9** | -0.072900000003533 | -0.072900000000000 |
| **1.0** | -0.000009997050657 | 0.000000000000000 |

**Table 2. OTM and Exact Solution for Example 2**

| t | $u_{OTM}$ | $u_{Exact}$ |
|---|---|---|
| **0.5** | 1.648721270700139 | 1.648721270700128 |
| **1.0** | 2.718281828459005 | 2.718281828459046 |
| **1.5** | 4.481689070337228 | 4.481689070338065 |
| **2.0** | 7.389056098932960 | 7.389056098930650 |
| **2.5** | 12.18249396072259 | 12.182493960703473 |
| **3.0** | 20.08553692327158 | 20.085536923187668 |
| **3.5** | 33.11545195838827 | 33.115451958692312 |
| **4.0** | 54.59815003142499 | 54.598150033144236 |
| **4.5** | 90.01713129899351 | 90.017131300521811 |
| **5.0** | 148.41315909661534 | 148.41315910257660 |

**Table 3. Absolute errors using OTM at n=15 for Example 1 and 2**

| t | Absolute errors for Example 1 | t | Absolute errors for Example 2 |
|---|---|---|---|
| **0.1** | 1.3000e-14 | 0.5 | 1.1102e-14 |
| **0.2** | 2.1770e-12 | 1.0 | 4.0856e-14 |
| **0.3** | 1.1779e-10 | 1.5 | 8.3755e-13 |
| **0.4** | 3.6062e-11 | 2.0 | 2.3092e-12 |
| **0.5** | 5.2000e-13 | 2.5 | 1.9117e-11 |
| **0.6** | 7.1026e-11 | 3.0 | 8.3911e-11 |
| **0.7** | 7.5600e-12 | 3.5 | 3.0404e-10 |
| **0.8** | 6.0680e-12 | 4.0 | 1.7192e-09 |
| **0.9** | 3.5330e-12 | 4.5 | 1.5283e-09 |

| | | | |
|---|---|---|---|
| **1.0** | 9.9971e-06 | 5.0 | 5.9612e-09 |

We have done these computations by OTM for $n$=15. In all computations, in the first step with Maple 18 we obtained the $\Phi$ and $A$ matrices from the section 4on Chebyshev polynomials and equation (5.5).Then we continued the methodas explained in the section on the Operational Tau method via Matlab R 2011b(7.13.5.564).

## CONCLUSION

In this paper, two examples of the FNFDEs have been solved successfully by the operational tau method on bases of Chebyshev polynomials. This method reduced the FNFDEs to a system of linear algebraic equations, including conversion of the delay parts of the desired FNFDEs to some operational matrices.

The solution obtained using the suggested method shows that this approach solves the known problem from Bhrawy and Alghamdi [8] effectively and with far exceeding precision.
From the rexamples considered here, it can be easily seen that our extension of the OTM obtains results as accurate as possible.

## Acknowledgments

We thank the anonymous referee for providing constructive comments and for helping us in that way to improve this paper.

## References

1. C. Lanczos. Trigonometric interpolation of empirical and analytic functions. J. Math. Physics, Vol. 17, 1938,123-199.
2. E.L. Ortiz, H. Samara. An operational approach to the tau method for the numerical solution of non-linear differential equations. Computing, Vol. 27, 1981, pp. 15-25.
3. K. M. Liu, C. K. Pan. The automatic solution to systems of ordinary differential equations by the tau method. Computer and Mathematics with

Applications,Vol. 38, 1999, pp. 197-210.
4. J. SedigiHafshejani, S. KarimiVanani, J.Esmaily. Operational tau approximation for neutral delay differential systems. Journal of Applied Science, 14, 2011, pp. 2585-2591.
5. K. B. Oldham, J.Spainer. The Fractional Calculus. Academic Press, New York/London, 1974.
6. I. Podlubny. Fractional differential equations. Academic Press, New York, 1999.
7. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. Theory and applications of fractional differential equations. North-holland Mathematics Studies. Elsevier, Amsterdam, 2006.
8. A.H. Bhrawy, M. A. Alghamdi. A shifted Jacobi Gauss collocation scheme for solving fractional neutral functional-differential equations. Advances in Mathematical Physics, Vol. 2014, 2014, pp. 1 - 8.
9. AytacArikoglu, Ibrahim Ozkol. Solution of fractional differential equations by using differential transform method. Chaos Solution & Fractals,Vol. 34, 2007, pp. 1473-1481.
10. S. KarimiVanani, A. Aminataei. Operational tau approximation for a general class of fractional integro-differential equations. Computational and Applied Mathematics,Vol. 30, 2011, pp. 655-674.
11. F. Ghoreishi, S. Yazdani. An extension of the spectral tau method for numerical solution of multi- order fractional equations with convergence analysis. Computers and Mathematics with Applications, Vol. 61, 2011, pp. 30-43.
12. A.H. Bhrawy, A.S. Alofi, S.S. Ezz-Eldien. A quadrature tau method for fractional differential equations with variable coefficients. Applied Mathematics Letters, Vol. 24, 2011, pp. 2146-2152.
13. M.Y. Rahimi-Ardabili, S. Shahmorad. Iterative numerical solution of non-linear integro-differential equations by the Tau method. Applied Mathematics and Computation, Vol. 193, 2007, pp. 514-522.

# On some stochastic nonlinear equations and the fractional Brownian motion

**Mahmoud M. El-Borai , Khairia El-Said El-Nadi**

m_m_elborai@yahoo.com , khairia_el_said@hotmail.com
Faculty of Science, Alexandria University, Alexandria, Egypt

**DOI :** http://dx.doi.org/10.22039/cjcme.2017.02

**Abstract** : Some stochastic nonlinear parabolic partial differential equations driven by fraction Brownian motion are considered. Initial-value problems for these equations are studied. Some properties of the solutions are given under suitable conditions and with Hurst parameter less than half.

***Keywords*:** Fractional parabolic stochastic partial differential equations, fractional calculus , fraction Brownian motion.

**AMS Subject Classifications:** 60G18, 60H05, 60H10.

## 1. Introduction

In this note stochastic partial differential equations of the form:

$$dv(x,t) = dB_H(t) + f(x,t,L_2u(x,t))dt, \tag{1.1}$$

are considered, where $0 < H < \frac{1}{2}$, $t > 0, x \in R^n$,

$$v(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - L_1u(x,t), \tag{1.2}$$

$$L_1 u = \sum_{|q|\leq 2m} a_q(x) D^q u, L_2 u = \sum_{|q|\leq 2m-1} b_q(x) D^q u,$$

$$D^q = D_1^{q_1} \ldots D_n^{q_n}, D_j = \frac{\partial}{\partial x_j}, 0 < \alpha < 1,$$

$R^n$ is the n-dimensional Euclidean space, $q = (q_1, \ldots, q_n)$ is an n-dimensional multi index $|q| = q_1 + \ldots + q_n$ , $B_H(t)$ is fractional Brownian motion with Hurst parameter $H \in [0, \frac{1}{2}]$ , $B_H(0) = E[B_H(t)] = 0$, for all $t \in R = (-\infty, \infty)$ and

$$E[B_H(t) B_H(s)] = \frac{1}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\}, s, t, \in R,$$

( $E[X]$ denotes the expectation of a random variable $X$ ).

If $H = \frac{1}{2}$, then $B_H(t)$ coincides with classical Brownian motion $B(t)$. For $H \neq \frac{1}{2}$ , $B_H(t)$ is not a semi martingale, so one cannot use the general theory of stochastic calculus for semi martingale on $B_H(t)$, (see[1],[2],[3]).

Denote by $K^*$ the linear operator defined on the set of all step functions to a subset of the set of all square integrable function $L_2[0,T]$, such that:

$$(K_H^* \varphi)(s) = K_H(t,s)\,\varphi(s) + \int_s^T [\varphi(r) - \varphi(s)] \frac{\partial K_H(r,s)}{\partial r} d\, r,$$

where

$$K_H(t,s) = \left(\Gamma(H+\frac{1}{2})\right)^{-1} (t-s)^{H-\frac{1}{2}} F(H-\frac{1}{2}, \frac{1}{2}-H, H+\frac{1}{2}, 1-\frac{t}{s}),$$

$\Gamma$ denotes the gamma function and F (a, b, c, z)is the Gauss hyper geometric function. The process $B_H$ has an integral representation:

$$B_H(t) = \int_0^t K_H(t,s)\, dB(s), \qquad (1.3)$$

where $B = \{B(t) : t \in [0,t]\}$ is the Brownian motion defined by

$$B(t) = B[(K_H^*)^{-1}(\chi_{[0,1]})], \qquad (1.4)$$

where $(\chi_{[0,1]})$ is the indicator function).

Let $f : R \to R$ such that $E[f^2(B_H(t))] < \infty$, then

where

$$\psi(t,\omega) = \left[\frac{\partial}{\partial x} E\{f(x + B_H(T-t)\}\right]_{x=B_H(T)},$$

see [1].

It is supposed that:

(1) All the coefficients $a_q, b_q$ satisfy a uniform Hölder condition on $R^n$,

(2) All the coefficients $a_q, b_q$ are bounded on $R^n$,

(3) The operator $\frac{\partial}{\partial t}-\sum_{|q|=2m} a_q(x)D^q$ is uniformly parabolic on $R^n$.

This means that

$$(-1)^{m-1}\sum_{|q|=2m} a_q(x)y^q \geq c\,|\,y\,|^{2m}, c>0,$$

for all $x,y\in R^n, y\neq(0,\ldots,0)$, where $y^q=y_1^{q_1}\ldots y_n^{q_n}, |\,y\,|^2=y_1^2+\ldots+y_n^2$ and c is a positive constant,

(4) The function f is continuous on $R^n\times[0,T]\times R$.

It is assumed that

$$u(x,0)=u_0(x),\frac{\partial u(x,0)}{\partial t}=u_1(x), \tag{1.5}$$

where $u_0,u_1$ are given sufficiently smooth bounded functions on $R^n$.

Without loss of generality, we can assume that $u_0(x)=u_1(x)=0$

In sections 2,3 the solution of the stochastic Cauchy problem (1.1),(1.5) is studied.

The fractional Brownian motion has many different impotant applications with amazing range. This amazing range makes fractional Brownian motion a very interesting object to study, (see [4-7]).

## 2. Formal solutions

The solution of equation (1.2) is formally given by:

$$v(x,t)=B_H(t)+\int_0^t f(x,\theta,L_2u(x,\theta)d\theta, \tag{2.1}$$

where

$$u(x,t)=\alpha\int_0^t\int_0^\infty\int_{R^n}\theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)G(x,\xi,(t-s)^\alpha\theta)v(\xi,s)d\xi\, d\theta\, ds, \qquad (2.2)$$

where $G$ is the fundamental solution of the parabolic equation:

$$\frac{\partial u(x,t)}{\partial t}=\sum_{|q|\le 2m}a_q(x)D^q u(x,t).$$

The function $G$ satisfies the following inequality:

$$|D^q G(x,\xi,t)|\le \gamma t^{c_1} exp[-c_2\rho], \qquad (2.3)$$

where

$$\rho=|x-\xi|^{m_1}\, t^{m_2}, m_1=\frac{2m}{2m-1},$$

$$m_2=-\frac{1}{2m-1}, c_1=-\frac{n+|q|}{2m},$$

$\gamma$ and $c_2$ are positive constants, [8-10]. The definition of the function $\zeta_\alpha(\theta)$ can be found in [8].

## 3. Fractional integral representation

Let $I_{a^+}^{\alpha}$ be the fractional integral operator defined by

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \alpha > 0.$$

Denote by $I_{a^+}^{\alpha}(L_2[a,b])$ the image of $L_2[a,b]$ by the operator $I_{a^+}^{\alpha}$. The operator $K_H$ on $L_2(0,T)$ associated with kernel $K_H(t,s)$ is an isomorphism from

$$L_2[0,T] \; onto \;\; I_{0^+}^{H+\frac{1}{2}}(L_2[0,T])$$

and it can be expressed in terms of fractional integrals by

$$(K_H \; g)(s) = I_{0^+}^{2H} s^{\frac{1}{2}-H} I_{0^+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} g,$$

$$(K_H \; g)(s) = \int_0^t K(t,s) f(s)ds.$$

The inverse operator $K_H^{-1}$ is given by

$$K_H^{-1} g = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0^+}^{2H} \; g,$$

for all $g \in I_{0^+}^{H+\frac{1}{2}}(L_2[0,T])$. If $g$ is absolutely continuous, it can be proved that

$$K_H^{-1} g = s^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} \; g', g' = \frac{dg}{ds}, \qquad (3.1)$$

where $D_{a^+}^{\alpha}$ is the fractional derivative defined by

$$D_{a^+}^{\alpha} \; g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{g(s)}{(t-s)^{\alpha}} ds,$$

see [3],[6]. A weak solution of equation (2.1) is defined by a couple of adapted processes $(B_H, v)$, for every fixed $x$ on a filtered probability space

($\Omega, F, P, \{F_t : t \in [0,T]\}$), such that

(a) $B_H$ is an $F_t$ - fractional Brownian motion,

(b) $v$ and $B_H$ satisfy (2.1).

Suppose that equation (2.5) has a weak solution. Then using the definitions of the operators $K_H$, $K_H^{-1}$ and the representation(1.1), one can write equation (2.1) in the form

$$v(x,t) = \int_0^t K_H(t,s) d\tilde{B}(x,s), \tag{3.2}$$

$$\tilde{B}(x,t) = B(t) + \int_0^t \eta(x,s) ds,$$

$$\eta(x,s) = K_H^{-1} g(x,.)(s),$$

$$g(x,\theta) = \int_0^\theta f(x,s,L_2 u(x,s)) ds.$$

**Theorem 3.1.** Let $H < \frac{1}{2}$ and $v$ be a weak solution of equation (2.5). If $f$ is a Borel function on $R^n \times [0,T] \times R$ and satisfies the linear growth condition

$$|\ f(x,t,u)| \leq C(1+|u|), \tag{3.3}$$

for all $u \in R, x \in R^n, t \in [0,T]$, (where C is a positive constant), then $g(x,.) \in I_{0+}^{H+\frac{1}{2}}(L_2[0,T])$.

**proof.** From (2.1), (2.2), (2.3) and (3.3) it can be deduced that

$$V(t) \leq |B_H(t)| + Ct + C_1 \int_0^t V(s)ds,$$

where $C_1 > 0$ is a constant and $V(t) = Sup_x |v(x,t)|$. The last inequality leads to

$$V(t) \leq |B_H(t)| + C_1 \int_0^t e^{C_1(t-\theta)} |B_H(\theta)| d\theta + C_2(e^{C_1 t} - 1). \tag{3.4}$$

Thus from (3.4) we get

$$\int_0^t V^2(s)ds \leq C_3 \int_0^t B_H^2(s)ds + C_4, \tag{3.5}$$

where $C_2 > 0, C_3 > 0$ are constants. From (3.3) and (3.5), we get

$$\int_0^T g^2(x,\theta)d\theta \leq C_4 T + C_5 \int_0^T B_H^2(s)ds + C_6, \tag{3.6}$$

where $C_4, C_5$ and $C_6$ are positive constants.

It is easy to see that

$$|I_0^{H+\frac{1}{2}}| = \frac{1}{\Gamma(\alpha)} |\int_0^t (t-s)^{H-\frac{1}{2}} g(x,s)ds|$$

$$\leq \frac{1}{\Gamma(\alpha)} (\int_0^t (t-s)^{2H-1} ds)^{\frac{1}{2}} (\int_0^t g^2(x,s)ds)^{\frac{1}{2}}. \qquad (3.7)$$

The required result follows from (3.5) and (3.6).

It is clear that $K_H^{-1} g(x,.) \in L_2[0,T]$ a.s. if and only if $g(x,.) \in I_{0+}^{H+\frac{1}{2}}(L_2[0,T])$ a.s.

Let $\zeta(x,T) = exp[-\int_0^T \eta(x,s)dB(s) - \frac{1}{2}\int_0^T \eta^2(x,s)ds]$.

If $f$ is bounded, then $\zeta(x,T)$ defines for every $x \in R^n$ a random variable such that the measure $\tilde{P}$ given by $d\tilde{P} = \zeta(x,T)dP$ is a probability measure equivalent to P. If $E\tilde{P}$ denotes the expectation with respect to $\tilde{P}$, then

$$E^{\tilde{P}}[\zeta(x,T)] = 1. \qquad (3.8)$$

From (3.1), (3.7), theorem (3.1) and Girsanov theorem, we see that $v$ is an $F_t$ -fractional Brownian motion with Hurst parameter H under the probability $\tilde{P}$, (see [7]).

**Lemma 3.1.** If $f$ is bounded ,then

$$E^P[\zeta^\alpha(x,T)] \leq exp[C|(2\alpha-1)(\alpha-1)|T],$$

where C is a positive constant.

**Proof.** We can deduce from the results in [ 7 ] that

$$E^{\tilde{P}} exp(-2\alpha\int_0^T \eta(x,s)dB(s) - 2\alpha^2\int_0^T \eta^2(x,s)ds) = 1,$$

for all $\alpha \in R$

Now

$$E^{\tilde{P}}[\zeta^{\alpha}(x,T)] = E^{\tilde{P}} exp\left[-\alpha\int_0^T \eta(x,s)dB(s) - \frac{\alpha}{2}\int_0^T \eta^2(x,s)ds\right]$$

$$\leq \left( E^{\tilde{P}} exp 2|\alpha^2 + \frac{\alpha}{2}|\int_0^T \eta^2(x,s)ds)^{\frac{1}{2}}\right)$$

On the other hand, using (3.1)

we get

$$|\eta(x,s)| = s^{H-\frac{1}{2}} I_{0^+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} f(x,s,L_2 u(x,s))$$

$$\leq \frac{M_1}{\Gamma(\frac{1}{2}-H)} s^{H-\frac{1}{2}} \int_0^s (s-\theta)^{-\frac{1}{2}-H} \theta^{\frac{1}{2}-H} d\theta,$$

where $M_1$ is a positive constant, $(|f| \leq M_1)$. Thus

$$E^{\tilde{P}} exp[2|\alpha^2 + \frac{\alpha}{2}|\int_0^T \eta^2(x,s)ds \leq exp[2|\alpha^2 + \frac{\alpha}{2}|M_2 T]],$$

where $M_2$ is a positive constant.

Using the fact that

$$E^{P}[\zeta^{\alpha}(x,T)] = E^{\tilde{P}}[\zeta^{\alpha-1}(x,T)],$$

we get the required result.

We can deduce from (3.1) that the operator $K_H^{-1}$ preserves the adaptability property. In other words the process $\eta(x,s)$ is adapted.

Let b be a positive Borel function defined on $[0,T]\times R$ such that the following integral.

$$\|b\|_{q,\gamma} = [\int_0^T (\int_R b^q(t,v)dv)^{\frac{\gamma}{q}} dt]^{\frac{1}{\gamma}}$$

exists, where $q > 1, \gamma > \frac{q}{q-H}$.

In this case we say that b belongs to $L_{q,\gamma}$, then by using lemma (3.1) the results of Naulart and Ouknine in [7] can be directly generalized to obtain the following estimations

$$E\int_0^T b(t,v(x,t))dt \le C\|b\|_{q,\gamma},$$

$$E\, exp[\int_0^T b(t,v(x,t))dt \le Q(\|b\|_{q,\gamma})],$$

where C is a positive constant and Q is a real analytic function, [11].

**Theorem 3.2**. If $f$ is continuous on $R^n\times[0,T]\times R$ and satisfies the Lipschitz condition;

$$|f(x,t,u)-f(x,t,v)|\le C|u-v|$$

for all $x\in R^n, t\in[0,T], u,v\in R$, where C is a positive constant, then there is weak solution v of equation (2.5). Moreover

$$E[v^2(x,t)] < \infty.$$

**Proof**. We shall use the method of successive approximations.

Set

$$v_{k+1}(x,t) = B_H(t) + \int_0^t f(x,\theta, L_2 u_k(x,\theta))d\theta,$$

$$u_k(x,t) = \alpha \int_0^t \int_0^\infty \int_{R^n} \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) G(x,\xi,(t-s)^\alpha \theta)) v_k(\xi,s) d\xi\, d\theta\, ds,$$

$$v_0(x,t) = 0.$$

Thus

$$|v_{k+1}(x,t) - v_k(x,t)| \le \frac{C^k}{(k-1)!} \int_0^t (t-\theta)^{k-1} |B_H(\theta)| d\theta.$$

it follows that the sequence $\{v_k\}$ uniformly converges with respect to $x$ to a stochastic process $v$. It is easy to see that

$$E[v^2(x,t)] \le \sum_{k=0}^\infty \frac{1}{(k+1)^2} \sum_{k=0}^\infty E[(k+1)^2 \{v_{k+1}(x,t) - v_k(x,t)\}^2].$$

This complete the proof of the theorem (see [10-21]).

## Acknowledgments

**We would like to thank the referees and Professor Dr. S.M. Ali Raeisian for their careful reading of the manuscript and their valuable comments.**

## References

[1] C. Bender: An $It\hat{o}$ formula for generalized functionals of fractional Brownian motion with arbitrary Hurst parameter. Stoch. Proc. and their Appl. 104 (2003), 81-106.

[2] F. Biagini, B. Oksendal, A. Sulem and N. Waller: An introduction to white noise theory and Malliavin calculus for fractional Brownian motion. The proceedings of the royal society 460 (2004), 347-372.

[3] L.Decreuse fond and A.S. $\ddot{V}$ stunel, Stochastic analysis of the fractional Brownian motion. Potential analysis 10 (1999), 177-214.

[4] Y. Hu and B. Oksendal: Functional white noise calculus and application to finance. Inf. Dim. Anal. Quantum prob.. Rel. Top. 6(2003) 1- 32.

[5] R. Elliott and J. Van Der Hock: A general fractional white noise theory anal applications to finance. Math. Finance 13 (2003), 301-330.

[6] S.G. Samko, A.A. Kilbas anal O.I. Mariachece, Fractional integrals and derivatives. Gordon and Breach Science, (1993).

[7] D. Nualart and Y. Ouknine: Regularization of differential equations by fractional noise Stoch. Proc. Appl. 102 (2002) 103-116.

[8] Mahmoud M. El-Borai, Some probability densities and fundamental solution of fractional evolution equations, Chaos,Soliton and Fractals,14(2002) 433-440.

[9] Khairia El-Said El- Nadi: On some stochastic parabolic differential equations in a Hilbert space. Journal of Appl. Math. and Stoch. Analysis, Vol.2,(2005),167-179.

[10] Mahmoud M. E-Borai, Khairia El-Said El- Nadi, O.L. Mostafa and Hamdy M. Ahmed, Volterra equations with fractional stochastic integrals, Mathematical problems in engineering, 2004: 5(2004) 453-468.

[11] Khairia El-Said El-Nadi, On some stochastic differential equations and fractional Brownian motion, Int. J. of Pure and Applied Math., Vol24, No.3, (2005) 415-425.

[12] Khairia El-Said El-Nadi, Asymptotic methods and difference fractional differential equations. Int. J. Contmp. Math. Sci. Vol.1, No.1 (2006) 3-14.

[13] W. Wyss: The fractional diffusion equation, Journal Math. Phys. 27 (1986), no 11, 2782 - 2785.

[14] Khairia El-Said El-Nadi, On some stochastic models of cancer, Canadian J. on Biomedical Engineering and technology, Vol.1, No.1 (2010), 42-48.

[15] A. Ju. Veretemikove: On strong solutions and explicit formulas for solutions of stochastic integral equations, Math USSR Sb. 39 (1981) 387-403.

[16] Mahmoud M. El-Borai, Khairia El-Said El-Nadi and Iman G. El-Akabawy, Approximate solutions for nonlinear fractional heat equations, Int. J. Contemp. Math. Sciences, Vol. 2, 2007. No.27, 1307-1316.

[17] Mahmoud M. El-Borai, Khairia El-Said El-Nadi and Iman G. El-Akabawy, On some fractional evolution equations, Computers and Mathematics with Applications, 59, (2010) 1352-1355.

[18] Mahmoud M. El-Borai, Khairia El-Said El-Nadi and Hoda A. Foad, On some fractional stochastic delay differential equations, Computers and Mathematics with Applications, 59, (2010), 1165-1170.

[19] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, Integrated semi groups and Cauchy problem for some fractional abstract differential equations, Life Science Journal, (2013), 10(3),793-795.

[20] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, An inverse fractional Cauchy problem, Life Science Journal, (2013), 10(3), 1705-1709.

[21] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, A prabolic transform and some stochastic ill-posed problems, British Journal of Mathematics and Computer Science, 9(5), (2015), 418-426.

# An Extremal Property of Delaunay Triangulation and Its Applications in Mathematical Physics

**Hayk S. Sukiasyan**
Institute of Mathematics, Academy of Sciences of Armenia
haik@instmath.sci.am

**AMS 2010 Mathematical Subject Classification** : 49J20, 52A40

***Keywords*:** Delaunay triangulation, finite elements method, Maxwell equations for magnetic field.

**DOI :** http://dx.doi.org/10.22039/cjcme.2017.03

**Abstract:** An extreme extreme property of the Delaunay triangulation is proved. Using this extreme property, the theorem is obtained, that the optimal mesh for the numerical solution of the Maxwell equation is the Delaunay triangulation.

## 1. Introduction

The popular method for the numerical solution of some problems of mathematical physics is the finite elements method. This method needs a mesh of triangles. The convergence rate of the process of numerical solution of the problem by the finite elements method depends on the geometrical configuration of the mesh.

In [2], [4] the Delaunay triangulation is recommended as triangle mesh for Maxwell equation for electromagnetic field.

In the present paper an extreme property of the Delaunay triangulation is proved. Using this extreme property, the theorem is obtained, that for the numerical solution of the Maxwell equation the optimal mesh is the Delaunay triangulation.

## 2. Delaunay Triangulation

Let $\{P_i\}_{i=1}^{n}$ be a finite set of points in the plane. A set $\{D_j\}_{j=1}^{m}$ of triangles is called the triangle mesh with knots $\{P_i\}_{i=1}^{n}$, if the following conditions are fulfilled:.

a) The interiors of triangles are pairwise disjoint;

b) The set of all vertices of triangles is the set $\{P_i\}_{i=1}^{n}$;

c) The union of triangles fills the whole of convex hull of the knots:

$$\bigcup_{j=1}^{m} D_j = conv\{P_i\}_{i=1}^{n}.$$

A triangle mesh $\{D_j\}_{j=1}^{m}$ is called Delaunay triangulation (see [1]) with knots $\{P_i\}_{i=1}^{n}$, if the following condition is fulfilled:

d) For any triangle $D_j$ $\operatorname{int} S(D_j) \cap \{P_i\}_{i=1}^{n} = 0$, j=1,…,m,

where *S(D)* is the circumscribing circle of triangle *D*.

Let T=$\{D_j\}_{j=1}^{m}$ be a triangle mesh with fixed set of knots $\{P_i\}_{i=1}^{n}$.

For a triangle *D* we denote by *A(D)* the set of interior angles of *D*. Consider the following expression depending on the mesh:

$$S(T) = \sum_{j=1}^{m} \sum_{\alpha \in A(D_j)} \cot\alpha. \qquad (2.1)$$

**Theorem 2.1.** The sum of cotangents (2.1) as a function on the meshes with fixed set of knots reaches his minimum for the Delaunay triangulation with the same set of knots.

**Proof.** First we prove the assertion of Theorem 2.1 for the meshes, those contain only two triangles. Consider a convex tetragon with vertices A, B, C, D, see Fig.1.

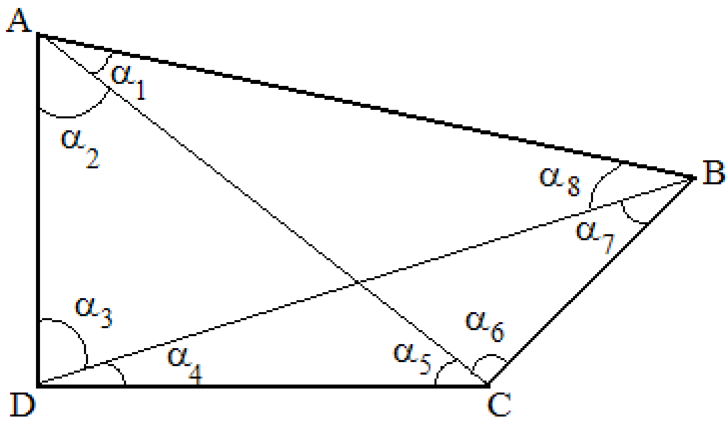



**Fig. 1.**

It is possible two triangle meshes with the knots A, B, C, D. The first mesh $T_{BD}$ with diagonal BD contains the triangles $\Delta$ABD and $\Delta$BCD. The second mesh $T_{AC}$ with diagonal AC contains the triangles $\Delta$ABC and $\Delta$ACD. For the mesh $T_{BD}$ the sum of cotangents is

$$S_{BD} = ctg(\alpha_1 + \alpha_2) + ctg\alpha_3 + ctg\alpha_8 + ctg(\alpha_5 + \alpha_6) + ctg\alpha_4 + ctg\alpha_7 \ .$$

For the mesh $T_{AC}$ we have

$$S_{AC} = ctg(\alpha_3 + \alpha_4) + ctg\alpha_2 + ctg\alpha_5 + ctg(\alpha_7 + \alpha_8) + ctg\alpha_1 + ctg\alpha_6 \ .$$

If

$$\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 > \pi,$$

then the point D lies outside of the circumscribing circle of triangle $\Delta$ABC and $S(T_{AC})<S(T_{BD})$. Hence the mesh $T_{AC}$ is a Delaunay triangulation. If

$$\alpha_1+\alpha_2+\alpha_5+\alpha_6<\pi,$$

then the point C lies outside of the circumscribing circle of triangle $\Delta$ABD and $S(T_{AC})>S(T_{BD})$. Hence the mesh $T_{BD}$ is a Delaunay triangulation. In both cases the assertion of Theorem 2.1 is valid.

Now consider a general finite set of points $\{P_i\}_{i=1}^n$, n>4. Among triangle meshes with the set of knots $\{P_i\}_{i=1}^n$ there is a mesh T with minimal sum of cotangents (2.1). Assume that the mesh T is not Delaunay triangulation. Then there exists a triangle $\Delta$ABC from the mesh T, such that the circumscribing circle of $\Delta$ABC contains a knot D from $\{P_i\}_{i=1}^n$. It is easy to see, that the point C lies outside of the circumscribing circle of triangle $\Delta$ABD. Therefore $S(T_{AC})>S(T_{BD})$. After the change of the diagonals AC and BD in the tetragon ABCD we obtain a new mesh T'. Observe that $T_{AC}\subset T$, while $T_{BD}\subset T'$. The inequality $S(T_{AC})>S(T_{BD})$ implies S(T)>S(T'), which contradicts the minimality of S(T). Theorem 2.1 is proved.

## 3. An Application to the Numerical Solution of Equations System

Let T=$\{D_j\}_{j=1}^m$ be a triangle mesh with the set of knots $\{P_i\}_{i=1}^n$. A knot $P_i$ of mesh T can be a boundary knot, which belongs to the convex hull of T, or an interior knot. Let *I(T)* be the set of interior knots of the mesh T. Denote by V(D) the set of vertices of triangle D, and by $E_i$ the set of triangles D from T connecting the knot $P_i$:

$$E_i=\{D_j\in T: P_i\in V(D_j)\},\quad i=1,2,...,n.$$

Consider the following system of linear equations with respect to unknowns $x_i$:

$$\sum_{D\in E_i}\ \sum_{v\in V(D)\cap I(T)} ctg\,\alpha(v,D)x_k = b_i,\quad P_i \in I(T),\quad i=1,...,N. \tag{3.1}$$

where $\alpha(v,D)$ is the interior angle of the triangle D at the vertex *v*, while *k* is the number of knot v, i.e. $v = P_k$ .

Observe that the system (3.1) depends on the mesh T. We would like to solve the system (3.1) numerically by the successive iterations method. The convergence rate of the successive iterations depends on the geometrical configuration of the mesh T.

The well-known principle of diagonal domination, roughly speaking, states that the fewer are the non-diagonal elements of matrix, the fewer iterations are needed for solution of the relevant system of linear equations. In the extreme cases when all non-diagonal elements vanish, we will need only one iteration for the solution.

We note, that the principle of diagonal domination is not law, and does not have any proof.

Using Theorem 2.1 and the principle of diagonal domination we can obtain the following assertion.

**Lemma 3.1.** Among triangle meshes with the fixed set of knots, the maximal convergence rate of the successive iterations to the numerical solution of the system (3.1) has the Delaunay triangulation with same set of knots.

## 4. Applications to the Differential Equations

We call *differential equation in the medium* a system of two two-dimensional differential equations

$$L(A(x,y), \mu(x,y)) = 0, \tag{4.1}$$

$$\mu = F\left( A, \frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial^2 A}{\partial x^2}, \ldots \right), \qquad (4.2)$$

where L is a differential operator acting on two depending variables A and $\mu$. The dependence F between the main variable (potential) A and the secondary variable (the medium function) $\mu$ is known. But the function F is complicate such that the substitution (4.2) in (4.1) reduces to a complicate equation, which is practically unsolvable, see [6], [7].

**Example 1**. The Maxwell equations for a magnetic field:

$$\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial A}{\partial x} + \frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial A}{\partial y} = \delta, \qquad (4.3)$$

$$\mu = P_6\left( \sqrt{\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial y}\right)^2} \right), \qquad (4.4)$$

where $A$ is the desired potential function, $\mu$ is the magnetic permeability of the medium, $P_6$ is a polynomial of 6 degree, $\delta$ is a given function, see [8], [10].

We consider a class of two-dimensional differential equations (4.1), (4.2) satisfying the following conditions:

(a) If $\mu$ is piecewise constant, then the equation (4.1) is solvable.

(b) If $A$ is piecewise linear, then $\mu$ is piecewise constant.

These conditions hold for the case, where the differential equations are reducible to a variation problem, i.e. there exists a functional J(A, $\mu$ (A)), which possesses an extremum on the solution of the differential equations (4.1), (4.2). Then (4.1) can be obtained from $J$ by the Euler variation formula.

The usual method for the numerical solution of such problems is the finite elements method. The considered domain is divided into small triangles (elements). The function $A$ is assumed linear within each triangle, and by condition (b) $\mu$ will be constant within each triangle. Then the search of the extremum of functional J is reduced to the solution of a system of linear equations in unknown values of potential $A$ at the vertices of triangles.

The usual scheme of solution is as follows. Starting from an initial piecewise constant $\mu_0$ we obtain by the condition (a) the piecewise linear function $A_1$. Then from $A_1$ we obtain by the condition (b) the piecewise constant function $\mu$ □. Then from $\mu$ □ □we get $A_2$ etc. If this process of the successive approximations converges, then the limit function $A = \lim_{n\to\infty} A_n$ is the desired (numerical) solution of the problem. Evidently, the equations (4.3), (4.4) satisfy the conditions (a) and (b).

**Proposition 4.1.** The convergence rate of the process of numerical solution of the problem (4.1), (4.2) by the finite elements method, depends on the geometrical configuration of the mesh, in the cells of which the medium factor $\mu$ is constant.

The application of the finite elements method to the equations (4.3), (4.4) reduces to the solution of following system of the linear equations (see [3], [5], [9]):

$$\sum_{D\in E_i}^{m} \sum_{v\in V(D)\cap I(T)} \cot\alpha(v,D)\mu(D)x_k = b_i, \qquad P_i \in I(T).$$

Using Lemma 3.1 we can obtain the following assertion.

**Theorem 4.1.** For the problem (4.3), (4.4) for any fixed knots set the best mesh is Delaunay triangulation.

Theorem 4.1. gives an opportunity to solve the problem (4.3), (4.4) using the variable mesh method. First, the problem is solved with the help of rough mesh with a small number of knots. Then, in the location where the error is at maximum, a new knot is added, and Delaunay triangulation is constructed with the added set of knots. The process continues till the error becomes sufficiently small.

**Example 2**. We solve the problem (4.3), (4.4) using the variable mesh method. The Figure 2 shows the solution, while the Figure 3 shows the final mesh.

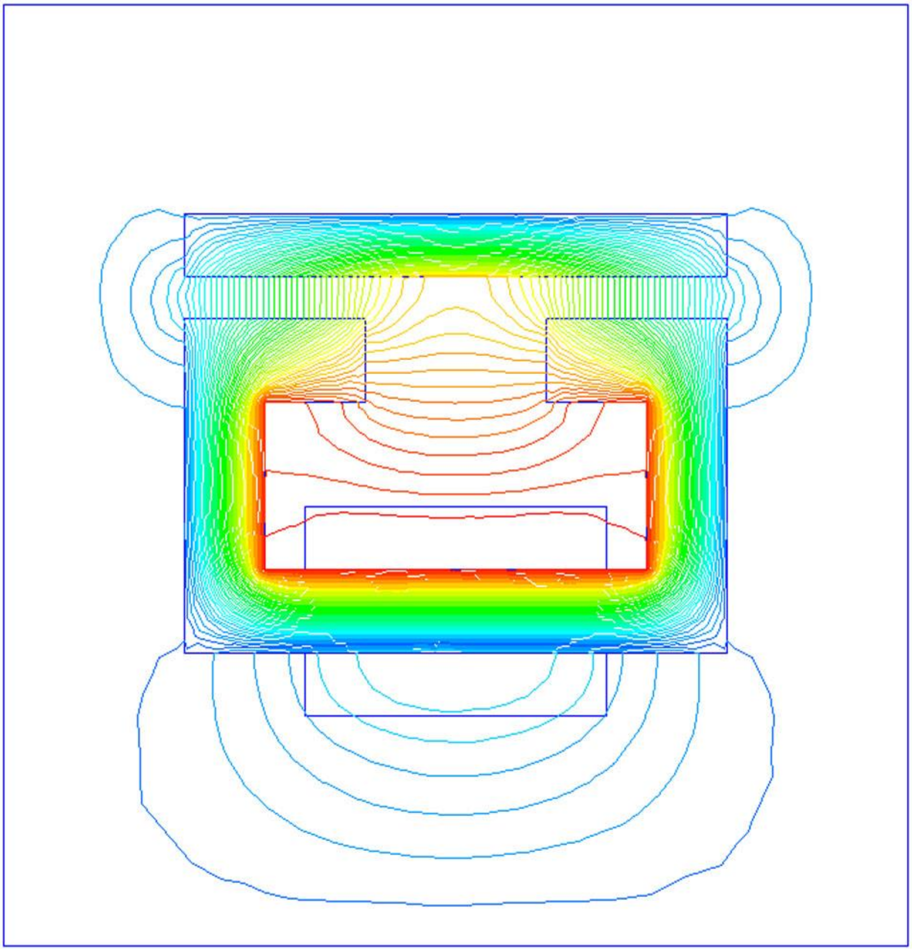


Fig.2

We can see in the figure 3 that the intensity of mesh knots is considerably different in the different sectors of the mesh, and corresponds to the density of the equipotential curves on the same sectors of the research domain.

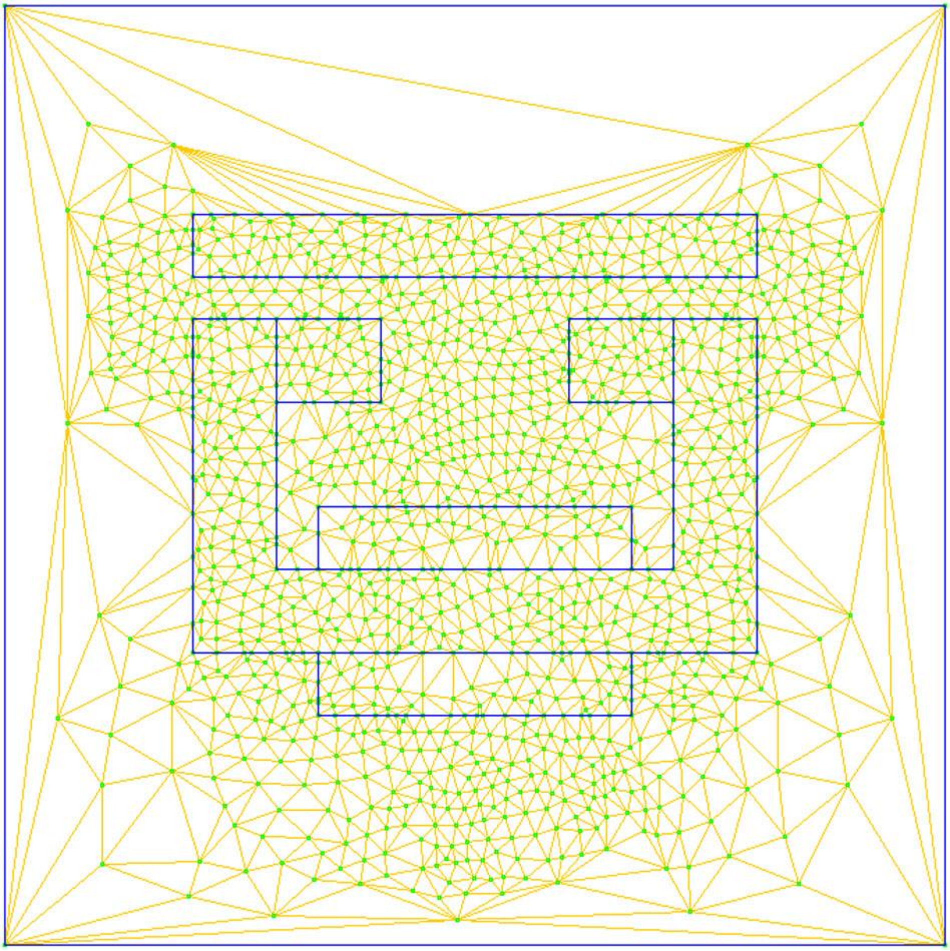


Fig.3

## Conclusions.

The sum of cotangents of interior angles as a function on the meshes with fixed set of knots reaches his minimum for Delaunay triangulation.

For any fixed knots set, for Maxwell equation of magnetic field the optimal mesh is Delaunay triangulation.

## References

1. Делоне Б.Н. О пустоте сферы. Изв. АН СССР. ОМЕН. - 1934.- №4. - С.793–800.
2. Ruppert J. A. Delaunay Refinement Algorithm for Quality 2-Dimensional Mesh Generation, NASA Ames Research Center, Journal of Algorithms, 1994.
3. Bobenko A. I., Springborn B. A. A discrete Laplace-Beltrami operator for simplicial surfaces, Discrete Comput. Geom., vol. 38(4), 2007, 740-756.
4. Glickenstein D. A monotonicity property for weighted Dealunay triangulations, Discrete Comput. Geom., vol. 38(4), 2007, 651-664.
5. Терзян А.А., Сукиасян Г.С., Пароникян А.Е. Об углах треугольной сетки для расчета магнитных полей методом конечных элементов. Известия НАН РА и ГИУА. Сер. ТН. - 2007. - Т. 60, №3. – С. 523-532.
6. Terzyan H., Simulation of Electromechanical Systems. Numerical Methods and Solutions, VDM Verlag, 2009, 280p.
7. Terzyan H.A., Tovmasyan N.E. Numerical methods of determining minima of a class of nonlinear functional. Theory of Functions and Applications. Collections of works dedicated to the memory of M.M.Djrbashian. -1995- Yerevan: Luoys.
8. Tovmasyan N. E. Non-Regular Differential Equations and Calculations of Electromagnetic Fields.World Scientific.Publishing Co. Ltd. Singapore, N.-J., London, Hong-Kong, 1998.
9. Terzyan H.A., Sukiasyan H.S., Akopyan A. E., Gevorgyan A. A. On construction of optimal mesh for computation of fields problems by finite elements method', Izv. NAN Armenii, ser. tekh. nauk, vol. 63, no. 3, 2010, pp. 319-327.
10. Silvester P., Chari M. Finite element solution of saturable magnetic field problems. IEEE Trans., PAS-89, 1970, v. 7, p. 1642=1648.

# On Numerical Comparison Between European and Asian Options

**Mohammad Ebrahim Alaei**

**Ph.d Student of Yerevan State University, Armenia**

m_e_alaei@yahoo.com

***Keywords:*** Black-Scholes formula, Asian option, jumping point
**DOI :** http://dx.doi.org/10.22039/cjcme.2017.04

**Abstract:** In this paper we show analysis on incomes or damages differences between two options for Black-Scholes formula, of course numerical comparison between European and Asian option in single option case about risk of price jumping at maturity.

## 1.Introduction

European option depends on maturity time. There exists a risk of price jumping at maturity. Asian option is more stable with respect to price jumping, because of the averaging feature, Asian options reduce the volatility inherent in the option.

For Asian options the payoff is determined by the average underlying price over some pre-set period of time. This is different from the case of the usual European option, where the payoff of the option contract depends on the price of the underlying instrument at exercise. Other advantage of Asian options is that these reduce the risk of market manipulation of the underlying instrument at maturity.

Let St be a commodity price at future time $t > 0$ and K is an arbitrary fixed positive constant. In this paper we consider the random variable St with lognormal distribution. Now by using the notation S(t)=St, the definition of call option for European option and Asian option are respectively:

$P_e(T)=E[S_t-K]_+$,

$P_a(T)=E[G(S_t)-K]_+$,

Where $X_+=\begin{cases}0, & X<0\\ X, & X\geq 0\end{cases}$ and $G(S_t)$ is average of $S(t_i)$ with i =1,2,...,m for example in geometric way. We assume that $G(S_t)=G(m,T)=[S(t_1).S(t_2)\ldots..S(t_m)]^{1/m}$, where $T=t_1+t_2+\ldots+t_m$ .The expectations $E[St - K]_+$ and $E[G(S_t)-K]_+$ are of interest for engineering, because they can be interpreted as call options of European and Asian derivatives.

The aim of the present paper is to compare of numerical results between European and Asian option in single option case about risk of price jumping at maturity. Our numerical comparison between European and Asian options is meant for engineering and financiers, and not for specialists of stochastic differential equations because we don't consider the solution of stochastic differential equations.

## 2. Related Definitions

European option income is $E(S_t - K)_+$ also Asian option income is $E(G(S_t)-K)_+$that we can use simple notation $I_1$ and $I_2$ for those respectively. Beside European option outcome (or damage) is $O_1 = (S_t - K)_+$ and Asian option outcome (or damage)is $O_2 = (G(S_t) - K)_+$ where $G(S_t)$ is the geometric average or $[S(t_1).S(t_2)\ldots S(t_m)]^{1/m}$.

Also we need an analysis on incomes and damages in this paper, so we can calculate incomes and damages for various initial times. We can get the graphics of incomes depending on initial time.

The "external points" of graphics can be interpret as moments of "price jumping". It is of "interest the cases" when, the maturity moment T is an "external point". Jumping point is not "local external points" at all. Jump is the maximum of the (absolute) different of price i.e. |p(k + 1) - p(k)| in a time segment. Consider the "(absolute) relative different" |p(k+1)-p(k)|/p(k) .To find the time moments, where the relative different is bigger than 0.03 (or 3 percent), jumping point is recognized.

## 3. Theoretical Part

If $S_0$ be a commodity price at today, $S_t$ be a commodity price at future time t > 0 and K is an arbitrary fixed positive constant (named "Strike price"). The main result of financial mathematics is called "Black-Scholes formula" that is described with below theorems for single options that they have been proved in Paper[1].

We include the proof of Theorem 2, because we can't found any elementary proof in other books, in particular the book [2] by J.C. Hull, contains this theorem without proof.

**Theorem 1.(for European Option)** Let $S_t$ be a commodity price,such that $\ln S_t$ has normal distribution with mean $\ln S_0 – \sigma^2 t/2$ and variation $\sigma^2 t$. Then

$P_e(T)=E(S_t - K)_+ = S_0\phi(d_1) - K\phi(d_2)$,

where:

$d_1=(\ln S_0 - \ln K+ \sigma^2 t/2)/(\sigma\sqrt{t})$

$d_2=(\ln S_0 - \ln K– \sigma^2 t/2)/(\sigma\sqrt{t})$

**Theorem 2.(for Asian Option)** Let S(t) be a commodity price, such that ln S(t) has normal distribution with mean $\ln S_0 - 2\ t/2$ and variation $\sigma^2 t$. Then call for Asian option is

$$E[G(S_t) - K]_+ = \exp\left( - \frac{m-1}{2m^2} T\sigma^2\right)\ S_0\phi(d_1) - K\ \phi(d_2)$$

where:

$d_1 = [\ln S_0 - \ln K+ \sigma^2 T/(2m^2).(2-m)][m/(\sigma\sqrt{t})]$

$d_2 = [\ln S_0 - \ln K– \sigma^2 T/(2m)][m/(\sigma\sqrt{t})]$

For Asian options is convenient the notation $S_t=S(t)$. The definition of call of Asian option is

$P_a(T) = E[G(S_t) – K]_+$

where $G(S_t)=G(m,T)$ is geometric average of $S(t_i)$ with i = 1, 2, ..., m, and $T = t_1 + t_2 + ... + t_m$.

**Proof :**We must remember proved Lemma 2 in [1] that describes: Let $\xi \sim N(\mu,\sigma 2)$. For any positive number K, the following equality holds:

$$E(e^{\xi}-K)_+= e^{\mu+\sigma^2/2} \cdot \phi\left(\frac{\mu+\sigma^2-\ln K}{\sigma}\right) – K \cdot \phi\left(\frac{\mu-\ln K}{\sigma}\right)$$

Now we have: $\ln G(m,T)=[\ln S(t_1)+\ln S(t_2)+...+\ln S(t_m)]/m$.

Since $\ln S(t) \sim N(\ln S_0-\sigma^2 t/2,\sigma^2 t)$ then $\ln[G(m,T)]\sim N(M,V^2)$

where $:M=\ln S_0-T.\ \sigma^2/(2m)$ (*)

and$V^2=T.\ \sigma^2/m^2$ (**)

Also we can apply the lemma 2 for $\xi=\ln[G(m,T)]$; So:

$$E[G(m,T)-K]_+=e^{M+V^2/2}.\phi[(M+V^2-\ln K)/V]-K.\phi[(M-\ln K)/V].$$

Finally by substituting (*) and (**) in above formula, we obtain $d_1$ and $d_2$ according to statements of Theorem2.

**Note I.** Consider the case when $S(t_1) = S(t_2) = ... = S(t_m)$, then Jump = 0 and $G(S_t) = S_t$ . In this case the difference between European and Asian options outcome is zero (Clearly); also the difference between European and Asian options income must be small.

***Proof:*** Consider the case $t_{j+1} = t_j + 1$. To prove this, consider the easy case m=2.

Thus $t_1 = t_0 + 1$, $t_2 = t_1+ 1 = t_0 + 2$ also in European and Asian option $T= t_0 + t$ and $T = t_1 + t_2 = 2t_0 + 3$ respectively (in simple case with $t_0 = 0$ we have: t = T in European one and T = 3 for Asian one).

Also:

$$I_1 = E(S_t - K)_+ = S_0\phi(d_1) – K\phi(d_2)$$

Where:

$d_1=(\ln S_0 - \ln K+ \sigma^2 T/2)/(\sigma\sqrt{T})$

$d_2=(\ln S_0 - \ln K- \sigma^2 T/2)/(\sigma\sqrt{Tt})$

Beside:

$I_2=E[G(S_t) - K]_+ = \exp( -\frac{\sigma^2 T}{8})\ S_0\phi(D_1 ) - K\ \phi(D_2)$

Where:

$D_1=(\ln S_0 - \ln K)[2/(\sigma\sqrt{t})]$

$D_2 = (\ln S_0 - \ln K- \sigma^2 T/4)[2/(\sigma\sqrt{t})]$

With $T = t_1 + t_2 = 2t_0 + 3$.

Now suppose that $\ln S_t$ has standard normal distribution N(0, 1) therefore the variance of distribution is $\sigma^2 t = 1$ and the mean of distribution is $\ln S_0 - \sigma^2 t /2 = 0$ i.e. $S_0 = e^{0.5}$

So: $d_1 = 1 - \ln k$ and$d_2 = - \ln K$. Then:

$I_1 = S_0\phi(d_1 ) – K\phi(d_2 ) = ... = e^{0.5}\ \phi(1 - \ln K) + K\phi(\ln K)$

Also for Asian one we have: $D_1 = 1 – 2\ln K$ and $D_2 = 1/2 - 2 \ln K$.

Then:

$I_2 = e^{- 1/8} .e^{1/2} .\phi(D_1 ) – K\phi(D_2 ) = ... = e^{3/8}\phi(1-2 \ln K)+K\phi( 1/2 -2 \ln K)$

By condition K = 1 we have:

$I_1 = e^{0.5}\ \phi(1) + \phi(0)$

$I_2 = e^{3/8}\phi(1) + \phi(1/2)$

We know: $\phi(0) = 0.5$, $\phi(1) = 0.8413$ and $\phi(0.5) = 0.6915$ therefore:

$I_2 - I_1 = 0.0285$ near 3 percent (This subject pointed in the section of related definitions).

**Note II.** Now if in Asian option m = 1, then Black-Scholes formula is the same European option.

T = t1:= t then $G(1, T) = [S(t_1)]^1 = S_t$ . Also by replacing m = 1 in formulas of Theorem.2 we have :

$P_a(T) = E(G(1, T) - K)_+ = E(S_t - K)_+ = \exp(-\frac{1-1}{2*1\hat{}2})S_0\phi(d_1) – K\phi(d_2)$ i.e.

$P_a(T) = E(S_t - K)_+ = S_0\phi(d_1) – K\phi(d_2)$

That is the same formula in European one, So $P_a(T)=P_e(T)$ for m=1. But for formulas of $d_1$and $d_2$ we have too:

$d_1 = [\ln S_0 - \ln K+ \sigma^2 t/(2*1^2).(2-1)][1/(\sigma\sqrt{t})]$

$d_2 = [\ln S_0 - \ln K– \sigma^2 t/(2*1)][1/(\sigma\sqrt{t})]$

Or:

$d_1 = (\ln S_0 - \ln K+ \sigma^2 t/2).[1/(\sigma\sqrt{t})]$

$d_2 = (\ln S_0 - \ln K– \sigma^2 t/2).[1/(\sigma\sqrt{t})]$

Where is the same relations in European option as seen in Theorem 1.

But if m increases, then the difference between European and Asian options increases or decreases? This question will study in the next sections with numerical results according to visual basic programming.

It is shown in Table.3 that the gap of incomes in Asian and European options will be bigger if m increases.

## Note III.

Also "zero Jump" implies "zero $|I_2 - I_1|$", using the continuity of Black-Scholes formula parameters, we conclude that "small Jump" implies "small $|I_2 - I_1|$". Similarly, if m is small, then the difference between European and Asian options is small, because if m=1, then the difference between European and Asian options is zero, also using the smoothness of Black-Scholes formula parameters.

## 4.Visual Basic Program

Using computer program (here with Visual Basic in Excel), we can calculate incomes and damages for various initial times.

Here we must calculate and compare the European and Asian options in the cases, when the maturity moment T is a "jumping point."

We consider some pair of European and Asian options with the sameparameters, particularly with the same maturity time moments T andcompare what option is better. It is important to notice that:

Income:$I_1=E[S_t - K]_+$ , Outcome:$O_1= [S_T - K]_+$ in European option,

and

Income:$I_2=E[G(S_t)-K]_+$ , Outcome:$O_2= [G(S_T)-K]_+$ in Asian option,

Also:

"Profit or Loss = Income – Outcome" in both of them.

## 5. Numerical Experiments

Suppose that $S_t=S(t)$ be real oil prices of one market and the first $t_0$be 20th price of the market (for computation of $\mu$ and $\sigma^2$)with strike price K=65,and t=10.

We know from the proof of Theorem.2: $\xi \sim N(\mu,\sigma 2)$ then $S_t = e^{\xi}$ has lognormal distribution or $\ln S_t \sim N(\ln S_0-\sigma^2 t/2,\sigma^2 t)$.Also K can be any positive number and $t_0$ and t are natural numbers so that $T=t_0+t$. Thus $\sigma^2$ can be estimated by: $\sigma^2$ =

$\frac{1}{n-1}\sum_{i=1}^{n} \ln\ S(t_{-i})$ here n=20 and $S(t_{-1}), S(t_{-2}),\ldots, S(t_{-20})$ are before prices of $t_0$ in the market.

Therefore we get the following numerical results within two tables. The first table for "European option" and second one for "Asian option" could compare incomes and damages of two formulas respectively.

Remember the columns of maturity $T = t_k$ , $S(T) := S_T$ , "price jumping" of $S(T)$: i.e. non relative Jumps = $|S(t_{k+1}) - S(t_k)|$, "Incomes:"$I_1 = E[S(t)-K]_+$ , $I_2 = E[G(S_t)-K]_+$ , "Outcomes": $O_1 = [S(T)-K]_+$ ,$O_2 = [G(S_T)-K]_+$and "Profit or Loss" are in the below tables:

**Table.1**

| T | S(T) | Jumps | $I_1$ | $O_1$ | Profit; Loss |
|---|---|---|---|---|---|
| 2008/09/12 | 103.36 | 0.44 | 52.70 | 38.36 | 14.34 |
| 2008/09/18 | 98.57 | 0.96 | 45.94 | 33.57 | 12.37 |
| 2008/09/26 | 106.85 | 0.80 | 38.37 | 41.85 | -3.48 |
| 2008/10/31 | 71.18 | 1.95 | 16.51 | 6.18 | 10.33 |
| 2008/11/10 | 66.79 | 1.18 | 11.16 | 1.79 | 9.37 |
| 2008/11/14 | 61.19 | 1.64 | 12.37 | 0.00 | 12.37 |
| 2008/05/20 | 130.16 | 3.97 | 51.34 | 65.16 | -13.82 |
| 2008/07/29 | 124.06 | 2.50 | 75.85 | 59.06 | 16.79 |
| 2008/10/13 | 83.63 | 3.41 | 32.67 | 18.63 | 14.04 |

**Table.2**

| T | $G(S_T)$ | Jumps | $I_2$ | $O_2$ | Profit; Loss |
|---|---|---|---|---|---|
| 2008/09/12 | 103.14 | 0.44 | 52.51 | 38.14 | 14.37 |
| 2008/09/18 | 98.09 | 0.96 | 45.79 | 33.09 | 12.70 |
| 2008/09/26 | 107.25 | 0.80 | 37.68 | 42.25 | -4.57 |
| 2008/10/31 | 70.20 | 1.95 | 11.28 | 5.20 | 6.08 |
| 2008/11/10 | 66.20 | 1.18 | 6.41 | 1.20 | 5.21 |
| 2008/11/14 | 62.00 | 1.64 | 8.52 | 0.00 | 8.52 |
| 2008/05/20 | 128.16 | 3.97 | 51.05 | 63.16 | -12.11 |
| 2008/07/29 | 125.30 | 2.50 | 75.55 | 60.30 | 15.25 |

| | | | | | |
|---|---|---|---|---|---|
| 2008/10/13 | 81.91 | 3.41 | 31.96 | 16.91 | 15.05 |

## 6. Conclusion

### 6.1 Comparison propositions with different jumps:

***Proposition I***: As we look in different cases between Table.1 andTable.2:

If Jumps < 1 then $|I_2 - I_1| < 5.2$

This subject was studied in Note III.

***Proposition II***: The different cases between Table.1 and Table.2 hasbeen prepared in Table.3 for m=2, m = 4, m = 6 and m = 10 there isbelow relationship for "Jumps" and "Absolute difference of outcomes":

If $|I_2 - I_1| > 6$ then Jumps > 1.17

Thus for positive number k we could be expected:

If $|I_2 - I_1| > k$ then Jumps > k/5

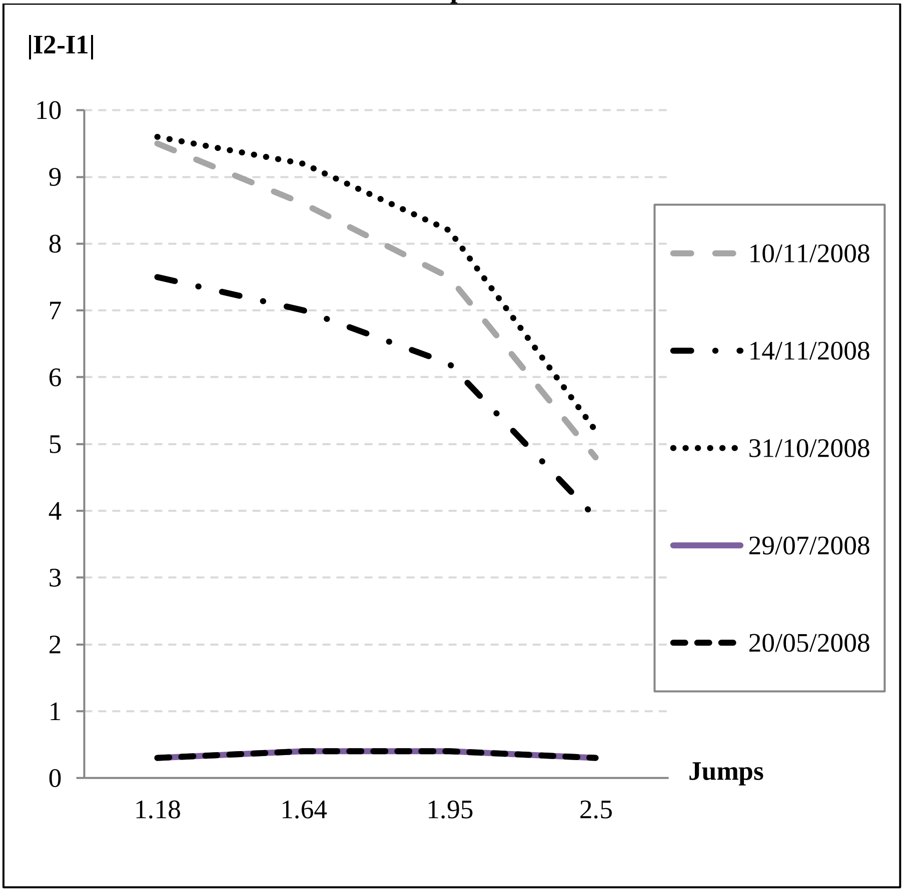
We can see this property and a result of Table.3 in Graph.1 .

**Table.3**($|I_2 - I_1|$ with various m)

| T | Jumps | m=2 | m=4 | m=6 | m=10 |
|---|---|---|---|---|---|
| 2008/05/20 | 3.97 | 0.3 | 0.4 | 0.4 | 0.3 |
| 2008/07/29 | 2.50 | 0.3 | 0.4 | 0.4 | 0.3 |

| | | | | | |
|---|---|---|---|---|---|
| 2008/10/31 | 1.95 | 9.6 | 9.2 | 8.2 | 5.2 |
| 2008/11/10 | 1.18 | 9.5 | 8.6 | 7.5 | 4.8 |
| 2008/11/14 | 1.64 | 7.5 | 7.0 | 6.2 | 3.9 |

**Graph.1**



## 6.2 Comparison propositions with various m :

***Propositions III:*** Now we want to compare numerical results for instance 2008/10/10 date. Table.4 has all parameters which used in Black-Scholes formula in European and Asian type with various m in Asian one.

**Table.4**

| Method | m | S(T) or $G(S_T)$ | $d_1$ | $d_2$ | $\phi$(d1 ) | $\phi$(d2 ) | Exp | I | O | Profit |
|---|---|---|---|---|---|---|---|---|---|---|
| European | 1 | 80.22 | 2.83 | 2.65 | 0.998 | 0.996 | - | 41.86 | 15.22 | 26.64 |
| Asian | 2 | 84.07 | 3.98 | 3.85 | 1 | 1 | 0.9922 | 41.02 | 19.07 | 21.95 |
| Asian | 4 | 86.55 | 5.86 | 5.78 | 1 | 1 | 0.9896 | 40.74 | 21.55 | 19.18 |
| Asian | 6 | 87.83 | 7.63 | 7.56 | 1 | 1 | 0.9898 | 40.76 | 22.83 | 17.93 |
| Asian | 10 | 91.81 | 11.52 | 11.48 | 1 | 1 | 0.9919 | 40.98 | 26.81 | 14.18 |

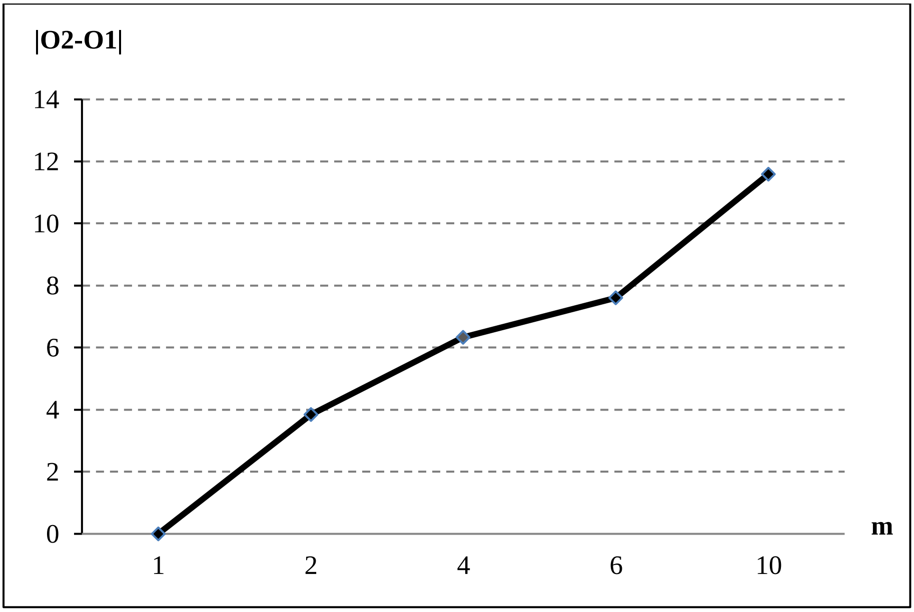
In Table.4; I and O are "Incomes" and "Outcomes" respectively for each methods with below commonly values:

$S_0 = 80.22$, $t = 10$, T = 2008/10/10, Jumps = 7.88, $K = 65$, $\mu = 4.658$ and $\sigma^2 = 0.003$.

Also in Table.4: Exp = $\exp(-\frac{m-1}{2m^2} T_i \sigma^2)$ for Asian method with $T_i = t_1 + t_2 + ... + t_m$.

As seen in Table.4, while m increases then difference between outcomes of two options is bigger too (Shown in Graph.2); but the difference between incomes of two options is nearly fix.

**Graph.2**



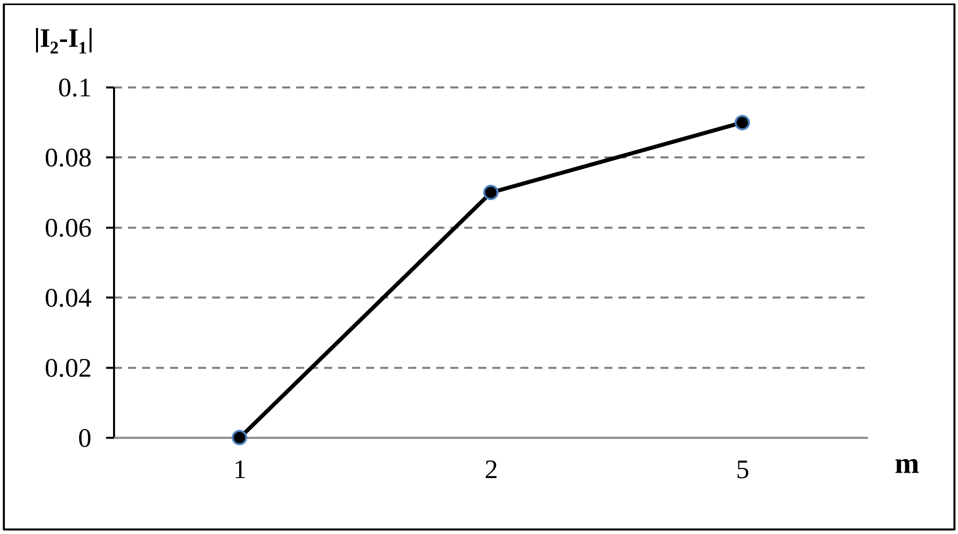

***Proposition IV:*** Now we want to study the case S(t) = S(t – 1) = ...=S(t - m + 1) where jump vanishes and G(m, t) = S(t). Thus formulas for Asian and European options compare together in Table.5.Notice that: "Incomes": $I_1 = E[S(t) - K]_+$ , $I_2 = E[G(S_t) – K]_+$; "Outcomes": $O_1 = [S(T) - K]_+$ , $O_2 = [G(S_T) - K]+$ (Case 1 for European and case 2 for Asian type).

**Table.5**

| T | Method | m | S(T) | $G(S_T)$ | Income | Outcome |
|---|---|---|---|---|---|---|
| 2007/08/03 | European | 1 | 72.49 | - | 6.80 | 7.49 |
| 2007/08/03 | Asian | 3 | 72.49 | 72.53 | 6.75 | 7.53 |
| 2007/08/03 | Asian | 4 | 72.49 | 72.44 | 6.75 | 7.44 |
| 2007/12/31 | European | 1 | 88.87 | - | 22.69 | 23.87 |
| 2007/12/31 | Asian | 2 | 88.87 | 88.85 | 22.62 | 23.85 |
| 2007/12/31 | Asian | 5 | 88.87 | 88.89 | 22.60 | 23.89 |
| 2008/11/10 | European | 1 | 66.79 | - | 11.16 | 1.79 |
| 2008/11/10 | Asian | 4 | 66.79 | 66.79 | 3.62 | 1.79 |

As seen in Table.5 the difference between outcomes of European and Asian is zero, according to Note I in Theoretical part. Also if m increases, then the difference between incomes of ones will be increase. This last subject is presented in Graph.3 for 2007/12/31.

**Graph.3**



**Acknowledgement.**The author wishes to thank Prof. Hayk Sukiasyanfor valuable and constructive comments.

## References

**1.** M.E. Alaei, Black-Scholes Formula for Multidimensional Asian Option,Armenian Journal of Mathematics (Registered in 2016).

**2.** John C. Hull, Options, Futures, and Other Derivatives, PrenticeHall, London, 2008.

**3.** John Vecer,Black-Scholes Representation for Asian Options, Frankfurt School of Finance and Management, 2012.

**4.** Jin E. Zhang,Theory of Continuously-Sampled Asian Option Pricing, City University of Hong Kong, 2000**.**

# Rahpoyan Andisheh & Honar cultural and artistic Institute

**Activities :**

- Film making
- Implementation of cultural & artistic projects
- Cultural and artistic trade
- Organizing to Perform concerts ,theaters , exhibitions
- Leading artistic lessons ( Painting , Graphic designing ,Photography ,…)
- Inviting artistic groups for performance

**Director** : Dr. S.M.Ali Raeisian

**Email** : rpandishehonar@gmail.com

**https://t.me/rpandishehonar**

## نحوه پذیرش مقالات

نشریه تخصصی CJCME دستاورد و نتایج تحقیقات اساتید و پژوهشگران رشته های فنی مهندسی و علوم پایه را منتشر می کند .

از محققانی که برای این نشریه مقاله تهیه می کنند درخواست می شود ضمن رعایت دقیق مفاد آئین نامه نگارش نشریه تخصصی CJCME ، مقالات خود را در یک نسخه فایل Word (حداکثر 20 صفحه) از طریق پست الکترونیک CJCMEmail@gmail.com ارسال دارند .

**تمامی مقالات توسط داوران ذیصلاح ارزشیابی می شوند** و نشریه تخصصی CJCME در پذیرش ، عدم پذیرش ، حذف و یا کوتاه نمودن مقالات برای چاپ آزاد است .

**فقط مقالاتی جهت انتشار در نشریه تخصصی CJCME مورد بررسی قرار می گیرند که قبلاً در نشریات علمی پژوهشی دیگر به چاپ نرسیده باشند .**

نویسندگان مقالات مسئول نوشته ها و نظرات خود هستند و آراء و نظریات آنان الزاماً نظر اعضای هیات تحریریه نیست .

**Professor Henk Koppelaar** , **Fac. Electric Engineering, Mathematics and Computer Science, Delft University of Technology Delft, The Netherlands**

*Fields of Interests :* *Neural Networks , Data Mining , AI , Automated Programming, Automated Discovery ,Computer Algebra, Discrete Mathematics, Experimental Mathematics, Fractional Calculus, Mathematical Physics, Numerical Analysis, Proof Automation .*

**Professor Vanya Alexandrovich Mirzoyan , Applied Mathematics and Informatics Faculty, National Polytechnic University of Armenia , Armenia**

*Fields of Interests :* *Geometry and Topology , Differential Geometry , Riemannian Geometry , Geometry of Riemannian Submanifolds*

**Dr. Antti Rasila** , **Aalto University, Helsinki, Finland**

*Fields of Interests :* *Complex Analysis, Partial Differential Equations, Computational Methods , Geometric function theory , Elliptic PDEs*

**Professor Masoud Sabbaghan , The College of Sciences of the University of Tehran , Iran**

*Fields of Interests :* *Fractal , Dynamical Systems , Fixed Point Theorem*

**Professor Hashem Saberi Najafi** , **The University of Guilan & Islamic Azad University of Lahijan branch , Iran**

*Fields of Interests :* *Numerical Analysis , Numerical Linear Algebra , Numerical Ordinary and Partial Differential Equations , Numerical Modeling of the Oceans*

اسم نشریه : Caspian Journal of Computational & Mathematical Engineering (CJCME)
دوره انتشار : دوفصلنامه
زبان نشریه : انگلیسی ، فارسی ، روسی ، ارمنی.
شمارگان : 500 نسخه
صاحب امتیاز : دکتر سید محمد علی رئیسیان
مدیر مسئول : دکتر سید محمد علی رئیسیان
رئیس هیت تحریریه : دکتر سید محمد علی رئیسیان
شماره مجوز نشریه : 75729

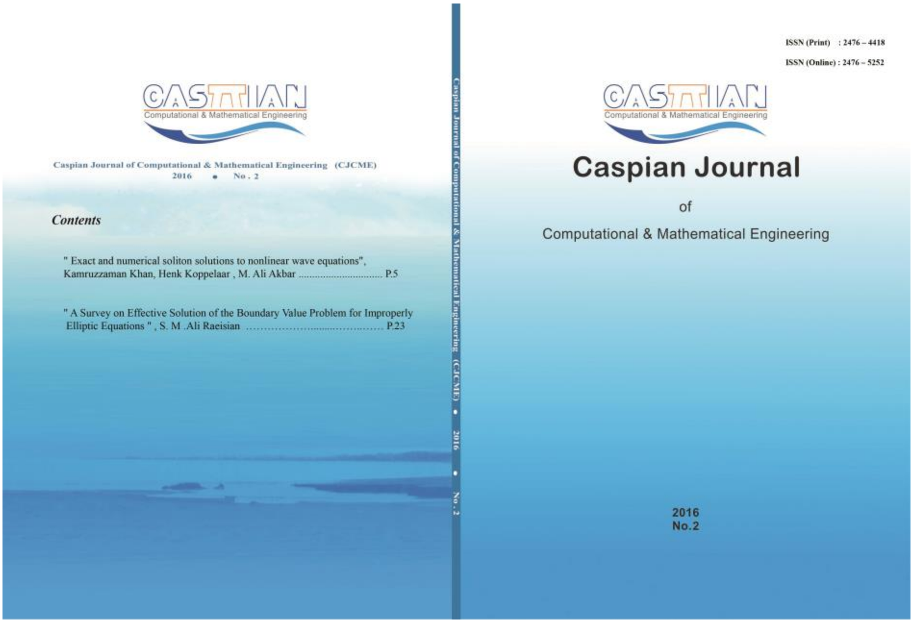
محل انتشار : رودسر
نشانی : گیلان ، رودسر ، صندوق پستی 1165 -44815 دکتر سید محمد علی رئیسیان
پست الکترونیک : smaraissian53@gmail.com , CJCMEmail@gmail.com
تماس : ( +98 )911 143 4988 , (+98 ) 13 - 42613646
وب سایت : https://caspianjcme.com

ویراستاری و صفحه بندی و تنظیم : موسسه فرهنگی و هنری ره پویان اندیشه و هنر
rpandishehonar@gmail.com

قیمت : 20000 تومان

## تاریخ انتشار : اسفند 1395


ISSN : 2476-4418
CASPIAN
Computational & Mathematical Engineering
Caspian Journal
of
Computational & Mathematical Engineering
2016
No.1
Caspian Journal of Computational & Mathematical Engineering (CJCME) • 2016 • No . 1
CASPIAN
Computational & Mathematical Engineering
Caspian Journal of Computational & Mathematical Engineering (CJCME)
2016 • No . 1
Contents
Defect Numbers of the Dirichlet Problem for Higher Order Partial Differential Equations in the Unit Disc , A. H. Babayan, V.A. Babayan ..... 4
A unified approach to the data fitting problem , L.Gevorgyan ..... 20
Introduction to phantom graph , Philip LLOYD ..... 31
A geometrical Isomorphism and the relativity of geometry , George Mpantes ..... 42
Convex geometric reasoning for crystalline energies , Thaicia Stona ..... 51


Caspian Journal of Computational & Mathematical Engineering (CJCME)
2016 • No . 2
Contents
" Exact and numerical soliton solutions to nonlinear wave equations",
Kamruzzaman Khan, Henk Koppelaar , M. Ali Akbar ............................ P.5
" A Survey on Effective Solution of the Boundary Value Problem for Improperly
Elliptic Equations " , S. M .Ali Raeisian ........................................ P.23
Caspian Journal of Computational & Mathematical Engineering (CJCME) • 2016 • No . 2
ISSN (Print) : 2476 – 4418
ISSN (Online) : 2476 – 5252
Caspian Journal
of
Computational & Mathematical Engineering
2016
No.2